Geometric Division

Fencepost division and replication problems

Figure 1 shows a fence panel in one-point perspective. A natural question arising in art is how to “halve” the fence: that is, how to draw the image of a line that divides the fence into two equal pieces in the real world. Of course, the image in the picture plane doesn’t preserve the equality of widths. So how do we draw the pictures below?

![Figure 1: Fence panel for problems 1–4.](image)

1. Divide a fence panel in half.
2. Divide a fence panel into thirds.
3. Describe a general method for dividing a fence panel into $n$ equal parts.
4. Continue the fence into the distance: that is determine where the rest of the fence posts go.

**Comment for the Mini-course participants:** The questions above will take way too long for a minicourse. These “Fencepost Problems” usually take 2–3 in-class days. We describe them and their solutions in greater detail in *Viewpoints: Mathematical Perspective and Fractal Geometry in Art*, so in this worksheet I am giving you just enough to get yourself oriented.

We strongly suggest trying the Fencepost division problems *yourself* when you have some free time: there are a half-dozen correct ways to come up with solutions to the first problem (not to mention many clever but incorrect solutions). Having students compare solutions (which is most general? which is most elegant? *etc*) has always been one of the mathematical high points of our semester. After doing these, a good art assignment is to have students design a word in 2-point perspective.
Euclidean proof relating fractions and slopes

There are many ways to do the division in the preceding section. In this section, we look at one of the more commonly discovered methods and prove that it works. Begin with a rectangle which is 1 horizontal unit by 1 vertical unit. (These units may or may not be equal: the rectangle might be one mile wide and one inch high. Still, from here forward, we will omit mentioning the units, and use only the fact that the rectangle is 1-by-1.)

5. Determine the slopes of the two dotted lines in Figure 2a.

6. Determine the slope of the bold dotted line $c_2$ in Figure 2b.

7. Determine the slope of the bold dotted line $c_3$ in Figure 2c.

8. Use induction to prove that the intersection of line $c_n$ and line $a$ can give us a fence panel that is $1/(n + 1)$ the width of the original fence panel.

Perspective proof relating slopes, vanishing points, and cross ratios

Usually, mathematics does not let us prove a theorem by demonstrating one specific example for which the theorem holds. We may not, for example, prove that the sum of two odd numbers is always an even number merely by noting that $3 + 5 = 8$.

But examples are sometimes enough to prove theorems in the realm of projective geometry applied to perspective art. One of the truly lovely aspects of projective invariants is that they allow us to prove some theorems not in their full abstract generality, but rather in one specific and very easy-to-compute circumstance. Then we can wave our magic projective wand and say, “so this always works!”

In the plan views below, we assume that the rectangle has one edge parallel to the picture plane.

9. In the plan view in Figure 3a, locate the vanishing points of the indicated lines. Name these points with appropriate capital letters (that is, $A$ is the vanishing point for line $a$, $B$ is the vanishing point for line $b$, etc.). Which one of these vanishing points is “at infinity”?

10. Determine the cross ratio $\times(ABCD)$. Explain why this cross ratio does not depend on the dimensions of this rectangle (in particular, it does not depend on the slope of the line $a$).
11. Now we do the same process for dividing a fence into fractions. In the Figure 3b, we assume that the left-most panel has $1/n$ the width of the entire panel. What is the slope of the line $c_n$ compared to that of the slope of line $a$?

12. Locate the vanishing point $C_n$ of the line $c_n$ in Figure 3b and determine the cross-ratio $\times(ABC_nD)$.

**Geometric Division Theorem**

11. Complete the statement of the following theorem: “Given a rectangle with edge vanishing points $B$ and $D$ and diagonal vanishing points $A$ and $C$. Draw a rectangle that has width $1/n$ of the original rectangle that has edge vanishing points _______ and _______ and primary points $A_n$ and $C_n$. Then the cross ratio $\times(ABC_nD) = ______$.

The proof of this theorem uses our specific instance (above), with the magic-wand fact that the cross ratio is a projective invariant. Done!

**Follow-up Art projects**

1. Take a photograph of something in the real world that contains regularly-spaced objects (or of something that is divided into a number of equal pieces). Photograph this object from two places: once from an angle so that the object is parallel to the picture plane, and another time so that
the object is in one-point perspective (one set of lines is parallel to the picture plane, and another vanishes straight ahead).

Print your pictures—you might want to lighten the image first, because you will be drawing on top of it. Verify that the first image contains collections of parallel, evenly spaced segments. (Measure those segments in the photo). Draw on the picture with first a pencil to test, and then with a dark marker when you’re sure of what you’re doing. In the second photograph, verify (first with pencil and ruler, then with dark marker and ruler) that the fencepost construction techniques we use in class work.

2. Write a word that is at least 4 letters long in 2-point perspective. The letters should (appear to) be a constant width, the spaces between them should be a (smaller) constant width, and the depth of the letters should be constant. The width of the lines in the letters should appear to be constant (the bar in a “T” is the same width as bars in an "H", for example). You can do this assignment in either of two formats:

Pencil-and-paper option: You should give the word a surrounding context (is it sitting on a table? mounted on the back wall of a room? In a vast plane with buildings on the horizon?) Draw lots of lines; draw neatly; use a straightedge.

Geogebra option: Your submission should include the main vanishing points as large, visible points. You will probably wish to hide most construction lines and other points, leaving only the word (and perhaps the horizon line) visible. We might use this version of the assignment later in the course when we get to shadows, so save it!

3. It is possible to create a Golden Rectangle (as in Figure 4) using ruler-and-compass techniques. Do a search in the library or on the web to discover how to do this on your own. Then, again using only ruler and compass, create a correct two-point-perspective image of a golden rectangle, beginning with the large square and adding on additional rectangles which you can then subdivide.

Figure 4: In this Golden Rectangle, each non-square rectangle is proportionate to the original rectangle.