Window Taping

The After Math

During a previous class, you and your classmates taped windows, making images of what the Art Director in each group saw outside the window. Your professor came around and made some comments, and then you took a photograph of another person’s window from as close to where that Art Director stood as possible. Today’s worksheet asks you to build on that work.

Here are some True/False questions about lines and planes in ℝ³ (that is, three dimensional Real space). As you proceed, you should work with your fellow students to justify your reasoning about these questions.

1. [T/F] If two distinct lines in ℝ³ are parallel, then they do not intersect.
2. [T/F] If two distinct lines in ℝ³ do not intersect, then they are parallel.
3. [T/F] If two distinct lines in ℝ³ are parallel, then they lie in the same plane.
4. [T/F] If two distinct lines in ℝ³ lie in the same plane, then they are parallel.
5. [T/F] If a plane in ℝ³ and a line not lying in that plane are parallel, then they do not intersect.
6. [T/F] If a line and a plane, both in ℝ³, do not intersect, then they are parallel.
7. [T/F] If the line ℓ is parallel to the plane π, then there is some line k ⊂ π that is parallel to ℓ.¹
8. [T/F] If there is some line k ⊂ π that is parallel to ℓ, then ℓ is parallel to π.

Now pull out the photograph you took. We will be looking at lines in the real world and at their images on the window (or even more specifically, the images of the tape in your photograph).

9. In the photograph you brought to class today, identify the images of a set of three lines that are parallel to each other in the real world and also parallel to the picture plane. Use your straightedge to extend the line segments to the edges of the paper. Are these images in your photograph parallel?

10. Now identify the images of a set of three of lines that are parallel to each other in the real world but not parallel to the picture plane. Use your straightedge to extend these line segments. Are these images in your photograph parallel?

11. In general, it is not surprising if pair of lines that are not parallel happen to intersect in a single point, but it is surprising if a set of three lines that are not parallel happen to intersect in a single point. Consider the lines from question 10. Describe how the intersection point you located in that question related to the plane of the window, to the original lines, and to the Art Director.

12. Formulate a conjecture:

¹In this class (as elsewhere in Projective Geometry), we use the convention of naming points with italicized capital letters, lines with italicized lower-case letters, and planes with lower-case Greek letters.
If a set of lines in \( \mathbb{R}^3 \) are parallel to each other and also parallel to the picture plane, then their images

If a set of lines in \( \mathbb{R}^3 \) are parallel to each other but not parallel to the picture plane, then their images

It’s time to start looking at how to address your conjecture. In this class, we will often analyze an image by using a plan view as in Figure 1. In this diagram, we see a viewer, the picture plane (which looks like a line in this diagram), and the object. In this plan view, the object is a beach towel lying on the horizontal ground.

![Figure 1: Two versions of a plan view: a side view and a top view.](image)

13. In the side view, the dot on the picture plane is the image of either the front corner or the back corner of the towel. Which corner is it? And how do you know?

14. Again on the side-view, use a straightedge to determine the edge of the other corner, and then shade in the image of the towel.

You discovered on the first day that looking through only one eye makes a huge difference. Rather than try to remember which eye our viewer looks through, when we draw top views, we’ll assume the viewer looks through his or her nose. (It’s silly, but it will make our diagrams much easier to draw!)

15. On the top view in Figure 1, draw the light rays connecting the corners of the towel to the viewer’s nose, and then draw the image of the towel.
What we have done so far is to look at points and their images. What can we say about lines and their images?

In Figure 2, we show a side view of the towel lying on the pebble-strewn ground. We will take advantage of this seemingly obvious fact: If you are looking at something, you can see it. If you are not looking at something, you can’t see it. In particular, if our viewer looks up at the sky (as we indicate in the plan view), she does not see the ground. (When we talk about looking in one direction, it helps to think of a “line of sight”, as though the viewer is looking through a straw and has no peripheral vision).

![Figure 2: Side view showing a viewer facing a towel on the pebble-strewn ground. When the viewer look up, she does not see the ground so she does not draw it.](image)

16. Using a straight-edge, draw the light rays connecting the viewer’s eye to the pebbles (you may want to draw these lightly).

   (a) As the pebbles get further and further away from the viewer, what can you say about the light rays?

   (b) Draw the images of the pebbles on the picture plane. As the pebbles get further from the viewer, what can you say about their images?

   (c) Suppose the line of pebbles goes on forever. Locate the point on the picture plane where the viewer goes from seeing the pebbles to not seeing the pebbles. Because the ground appears to vanish at this point on the picture plane, we call this point the vanishing point of the line of pebbles.

   (d) Draw the line of sight from the viewer to the vanishing point. How does this line relate to the line of pebbles?

17. Let us repeat this procedure with the four lines in Figure 3. Add some pebbles to the lines that extend the edge of the towel, and then draw their images.

   (a) Locate the vanishing points of these four lines. How many vanishing points are there?

   (b) Draw the line(s) that connect the viewer’s nose to the vanishing point(s). What can you say about how the line(s) relate to the edges of the towel?

18. On a separate sheet of paper, draw your own plan views, both side view and top view, showing a viewer, the picture plane, and two horizontal lines that are parallel to the picture plane. Answer these questions about your plan view:

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Figure 3: Top view showing a viewer looking at a towel, with the edges of the towel extended.

(a) Must the two lines you draw be parallel to one another?
(b) Locate the image of each line in the side view.
(c) Locate the image of each line in the top view.
(d) How does the image of each line relate to the original line?
(e) How many vanishing points do each of these lines have?

19. Compare your conjecture from question 12 to your answers to questions 16–18. Do your answers agree with one another?

20. Complete the following set of statements.

If a line is not parallel to the picture plane, then it has a vanishing point. That vanishing point is located _________________.

A collection of lines that are parallel to each other and also parallel to the picture plane has exactly _______ vanishing point(s).

A collection of lines that are parallel to each other but not parallel to the picture plane has exactly _______ vanishing point(s).

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**Homework**

**Questions**

1. Consider Figure 4, which shows two sketches of hallways. For each sketch, determine whether the artist was sitting or standing, and explain how you know.
2. Suppose a red door and a blue door, two doors along the same wall, are exactly $X$ units apart. Both Ryan and Barb are on the other side of a window, looking down the hallway at these doors, but Ryan is $X$ units further from the window than Barb. That is, the order is

Ryan - Barb - window - Red - Blue.

Each of the two artists creates a picture of the doors on the window as s/he sees them. Is the image of the red door (according to Ryan) the same size as the image of the blue door (according to Barb)? Use a plan view to answer this question. [This question was suggested by Sydney Seydel, F&M '13.]

3. more forthcoming

Figure 4: Two sketches of hallways for homework question 1.

Art Assignment

A1. Sketch your hallway in one-point perspective. Include at least two doorways, the floor and ceiling, and at least one “thing” on a wall. Make sure that lines that are supposed to be vertical in your sketch are vertical (similarly for lines that should be horizontal), and that lines that should go to the vanishing point do. Pay special attention to small details (e.g. door jambs and door knobs). Draw all lines neatly with a ruler. This sketch should take one-to-two hours.

Proof/Counterexample

For your the write-up of your proof(s) or counter-example(s) to the statements below, you should include and refer to a plan view.

1. If an artist draws an image of a hallway on a vertical canvas, doors that are further along the hall (and therefore further from the artist) will have smaller images on the canvas.

2. If an artist is draws an image of a building on a vertical canvas, windows that are directly above one another on higher floors (and therefore further from the artist) will have smaller images on the canvas.
Instructors’ notes for “Window Taping: The After Math”

Materials needed:
• pencil,
• straightedge,
• photographs of the window-taping exercise (see questions 9–12).

Lessons of the worksheet:
• basic understanding of parallel lines and planes;
• images of lines in a picture plane;
• plan views;
• definition of vanishing point;
• importance of the notion of “parallel” in determining the existence and location of the van-
  ishing point.

In the class before this one, you should take your students to tape windows. See Viewpoints: Mathematical Perspective and Fractal Geometry in Art for more detail on how to do this.

At some point during the exercise, you should ask students to take a break and wander to look at each other’s windows. They should try to take a picture of at least one window that is not theirs, and they should try to do in such a way that the tape lines up with the outside world. They will bring a printout of this photo to class for the “After Math”.

After they have taken photos, you should also prepare the students for this worksheet by pointing out to the groups that they drew sets of parallel lines in different ways. Some collections of parallel lines (those parallel to the window, meaning that if you extend the lines and the window forever, they wouldn’t intersect) have taped images that are parallel. Other collections of parallel lines (those not parallel to the window, meaning that if you extend the lines and the window, the lines would pierce the window) have taped images that are not parallel. You should extend these lines and show that (miraculously!) they all meet in one point. The students might even point out to you that the point has something to do with the art director.

The students will get to re-learn these lessons for themselves in a more rigorous way as they work through this “Window Taping: The After Math” worksheet.

Follow-up to the hallway sketching assignment

On the day the students turn in their hallway sketches, I usually begin this class by having students tape their just-completed hallway sketches on the wall around the classroom. We all get to view one another’s work, and I’ll do a gentle critique (pointing out and praising risks that students took, pointing out and cautioning students against common mistakes). I make these comments mostly general, as in “you’ll notice that some students had a lot more lines than other students, and those drawings seem even more realistic,” or “several people had trouble making the base of the door jambs horizontal. That’s an easy mistake to make, but you should guard against it.”
After the critique (and possibly another round of viewing), I ask students to come up with a list of questions they had while drawing or while looking at these pictures. Typical questions are

1. How do you know where to put the vanishing point?
2. How do you draw a door jamb or window sill correctly?
3. How high do you make the doors?
4. How wide do you make the doors?
5. How do you space the doors/ceiling tiles/etc. as they go back into the distance?
6. How do you know if the ceiling tiles are squares?

Some of these questions students ask (1, 2) will allow you to review material from the previous class. A few questions (3) will give you the chance to quickly introduce the notion of proportion. Many others (4, 5, 6) are difficult—I acknowledge these as very good questions and promise we’ll get to them later in the class. I try to remember which students asked them, so that at the appropriate time, I can say, “today we’ll be figuring out how to answer Sam’s question about ceiling tiles.”