Frame Diagonalization of Matrices

Fumiko Futamura

Mathematics and Computer Science Department
Southwestern University
1001 E University Ave
Georgetown, Texas 78626 U.S.A.
Phone: +1 (512) 863-1981
Fax: +1 (512) 863-5788

Abstract

This paper introduces a concept of diagonalization that uses not a basis of eigenvectors, but a frame. A frame is a generalization of a basis which is used in a number of signal and image processing applications. We first investigate the properties of frame diagonalization, drawing parallels with those of basis diagonalization. We then describe several methods of constructing frames for frame diagonalization. In particular, we prove the existence of a universal diagonalizer for each $n \in \mathbb{N}$ that simultaneously diagonalizes all matrices in $M_n(\mathbb{C})$, and create a method of frame diagonalization that works for any matrix in $M_n(\mathbb{C})$, uses at most $\lfloor 3n/2 \rfloor$ frame vectors and retains information about the eigenvalues of the matrix.

Keywords: diagonalization, frame, frame multiplier
2000 MSC: 42C15, 65F30, 65F15

1. Introduction

In nearly all diagonalization methods (notably with the exclusion of singular value decomposition), a matrix $A \in M_n(\mathbb{C})$ is diagonalizable if it can be written in the form $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix and the columns of S form a basis and the columns of $(S^{-1})^*$ form its biorthogonal dual. A generalization of a basis is a concept called a frame, which is a possibly overcomplete set of vectors which spans the space. Precisely because

Email address: futamurf@southwestern.edu (Fumiko Futamura)

Preprint submitted to Linear Algebra and Its Applications *October 21, 2011*

of their overcompleteness, frames are useful in a number of applications, in such areas as sampling theory, wavelet theory, signal processing and image processing [3, 4, 7]. Like bases, frames have duals, athough these duals may not be unique. What if, instead of diagonalizing a matrix using a basis and its dual, we use a frame and its dual? We describe this notion and investigate some interesting properties of this kind of diagonalization.

The notion of frame diagonalization has been studied to some extent in a more general setting which includes infinite dimensions [1, 2, 5]. In [2], Balazs introduces the notion of a frame multiplier which most closely resembles frame diagonalization, defined to be an operator acting on a Hilbert space $\mathcal{H}$ given by $M(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k$, where $(m_k)_{k \in K} \in \ell^\infty(K)$ and $(\psi_k)_{k \in K}$ and $(\phi_k)_{k \in K}$ are two frames for $\mathcal{H}$. In fact, if we let $\mathcal{H} = \mathbb{C}^n$ and let (ϕ_k) be a dual of (ψ_k), then we have the notion of frame diagonalization as described in this paper. The main focus of [1, 2, 5] is on properties of these multipliers based on properties of their coefficients, such as compactness and boundedness, which is relevant only in the infinite dimensional setting. In [2] however, Balazs considers the finite dimensional situation by introducing an algorithm for finding the best approximation of a matrix by a frame multiplier associated to a given pair of frames. In this paper, we take a different approach; instead of fixing a pair of frames and approximating by a frame multiplier, we give several algorithms for constructing a frame and its dual which will exactly frame diagonalize a given matrix. Through this approach, we find that in finite dimensions, we can always find a frame and a dual which will frame diagonalize a given matrix. In fact, we show several algorithms for constructing a number of frames and their duals which will frame diagonalize a given matrix. Many interesting questions arise through this approach. In particular, a question only partially addressed in this paper is, given a matrix, can we find a diagonalizing frame-dual pair satisfying particular properties? Given a matrix, can we characterize all possible frames and duals which frame diagonalize that matrix?

In section 2, we describe known diagonalization schemes, including unitary diagonalization and singular value decomposition, as well as the necessary background on frame theory. In section 3, we introduce the notion of frame diagonalization for finite matrices and explore some basic properties. In the last three sections, we create several algorithms for finding frames which diagonalize a given matrix. In section 4, we prove that there exists a universal frame and dual with n^2 elements to frame diagonalize any matrix in $M_n(\mathbb{C})$. This is very interesting, but perhaps not very useful since the diagonal matrix would have n^2 entries down the diagonal, and the original matrix itself has n^2 entries. Can we get away with fewer frame elements? It turns out that we can, although not universally; in section 5, we show that by using singular value decomposition, we can always frame diagonalize a matrix with $2n$ frame elements. We can do even better; in section 6, by using the Jordan canonical form, we show that we can frame diagonalize a matrix with at most $\lfloor 3n/2 \rfloor$ frame elements, specifically $n+k$ elements where k is the number of non-trivial Jordan blocks in the Jordan canonical form of the matrix. Furthermore, this algorithm creates a diagonal matrix that retains information about the eigenvalues of the original matrix.

2. Theory and Background

In this section, we include the common ways in which we can diagonalize a matrix in some sense. For additional background, refer to [8]. We also give the necessary background of finite dimensional frame theory, which sets the framework for the definition of frame diagonalization.

Definition 2.1. A matrix $A \in M_n(\mathbb{C})$ is *diagonalizable* if it is similar to a diagonal matrix, i.e., there exists a nonsingular matrix S and diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

The columns of the nonsingular matrix S form a basis, in particular, since $AS = S\Lambda$, a basis of eigenvectors of A. In fact, this is an alternate definition for diagonalization: A is *diagonalizable* if and only if its columns form a basis of eigenvectors. The diagonal entries of Λ are the corresponding eigenvalues.

In the special case where the eigenvectors form an orthonormal (orthogonal) basis, then the eigenvectors form the columns of a unitary (orthogonal) matrix and we say that A is unitarily (orthogonally) diagonalizable.

Definition 2.2. A matrix $A \in M_n(\mathbb{C})$ is *unitarily (orthogonally) diagonalizable* if it is unitarily (orthogonally) similar to a diagonal matrix, i.e., there exists a unitary (orthogonal) matrix U and diagonal matrix Λ such that $A = U\Lambda U^*$.

We know exactly the structure of unitarily diagonalizable matrices through spectral theorem.

Theorem 2.3 (Spectral theorem for normal matrices)**.** *$A \in M_n(\mathbb{C})$ is normal if and only if A is unitarily diagonalizable.*

So clearly, not all matrices are unitarily diagonalizable. In addition, not all matrices are diagonalizable as shown through the simple example below.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. As a triangular matrix, we see that the only eigenvalue of A is $\lambda = 1$ with an algebraic multiplicity of 2. So to be diagonalizable, this eigenvalue must have a geometric multiplicity equal to its algebraic multiplicity. However, it is easy to verify that it has a geometric multiplicity of 1. Hence A does not have a basis of eigenvectors and thus A is not diagonalizable.

We have been restricting ourselves to square matrices. If we consider $n \times m$ matrices, singular value decomposition provides a way of diagonalizing any of these matrices. With this method, the entries along the diagonal matrix are generally singular values, not eigenvalues. However, if A is a normal, positive semi-definite matrix, then its singular value decomposition coincides with its decomposition by a unitary matrix.

Theorem 2.4 (Singular value decomposition)**.** *Any $n \times m$ complex matrix A can be diagonalized in the sense that it can be written $A = U\Lambda V^*$, where*

- *U is the $n \times n$ unitary matrix whose columns are the eigenvectors of AA^*;*
- *V is the $m \times m$ unitary matrix whose columns are the eigenvectors of A^*A;*
- *Λ is the $n \times m$ diagonal matrix whose main diagonal consists of square roots of eigenvalues of AA^* and A^*A.*

We now turn to frames. We provide only the basic, necessary background of finite dimensional frame theory to understand frame diagonalization; for further details and additional background, refer to [3].

Definition 2.5. A finite sequence of vectors $(f_i)_{i=1}^m \subset \mathbb{C}^n$ is a *frame* for $\mathbb{C}^n$ if it is a spanning set for $\mathbb{C}^n$.

Though the definition of a frame is more involved in infinite dimensions, in finite dimensions, we can say that a frame is merely a spanning set which can be overcomplete. Notice, if the vectors in the frame are linearly independent, then $m = n$ and they form basis. In this way, frames can be thought of as a generalization of the concept of a basis. The advantages of overcompleteness are apparent when dealing with error-stricken data, or data with erasures [4, 6].

If we compile these frame vectors as columns of a matrix, we have an $n \times m$ matrix called a frame matrix.

Definition 2.6. Let $\mathcal{F} = (f_i)_{i=1}^m \subset \mathbb{C}^n$ be a frame for $\mathbb{C}^n$. The *frame matrix* F associated to $\mathcal{F}$ is an $n \times m$ matrix whose columns are the frame vectors:

$$F = [f_1 \, f_2 \, ... \, f_m].$$

Proposition 2.7. *An $n \times m$ matrix is a frame matrix if and only if its rows are linearly independent.*

For the proof, see [3]. What the proposition above tells us is that we can always add to the frame matrix $m - n$ linearly independent rows to form a nonsingular matrix, call it F', whose columns form a basis for $\mathbb{C}^m$. The columns of $((F')^{-1})^* = \tilde{F}'$ form the dual basis. Thus, if we remove the corresponding $m - n$ linearly independent rows from $\tilde{F}'$, we end up with an $n \times m$ matrix $\tilde{F}$ such that $\tilde{F}F^* = F\tilde{F}^* = I_n$. Since the rows of $\tilde{F}$ are still linearly independent, this matrix is a frame matrix, which we call a dual frame matrix for F. Since there are many ways of adding $m - n > 0$ linearly independent rows to form an invertible matrix, there are many possible dual frame matrices for F, and thus many possible duals for a given overcomplete frame.

Definition 2.8. An $n \times m$ matrix $\tilde{F}$ is a *dual frame matrix* for F if and only if $\tilde{F}F^* = F\tilde{F}^* = I_n$.

Re-translating this back into the language of vectors, we have the more traditional definition of a dual frame.

Definition 2.9. Let $(f_i)_{i=1}^m \subset \mathbb{C}^n$ be a frame for $\mathbb{C}^n$. A finite sequence of vectors $(\tilde{f}_i)_{i=1}^m \subset \mathbb{C}^n$ is a dual for this frame if for all $v \in \mathbb{C}^n$,

$$v = \sum_{i=1}^{m} \langle v, f_i \rangle \tilde{f}_i = \sum_{i=1}^{m} \langle v, \tilde{f}_i \rangle f_i.$$

3. Frame Diagonalization

Not all matrices can be digonalized, and although all matrices can be "diagonalized" using singular value decomposition, we lose the essential similarity structure in that $UV^* \neq I$ in general for a matrix with decomposition $A = U\Lambda V^*$. Here, we introduce an alternate diagonalization method, frame diagonalization, that not only allows for the "diagonalization" of any square matrix A in the sense that $A = \tilde{F}\Lambda F^*$ where Λ is a diagonal matrix, but also $\tilde{F}F^* = I$.

Definition 3.1. A matrix $A \in M_n(\mathbb{C})$ is *frame diagonalizable* by the $n \times m$ frame-dual matrix pair $(F, \tilde{F})$ if and only if $A = \tilde{F}\Lambda F^*$, where Λ is an $m \times m$ diagonal matrix. Thus for all $v \in \mathbb{C}^n$, $Av = \sum_{i=1}^m \langle v, f_i\rangle \lambda_i \tilde{f}_i$.

Notice, if the columns of F forms a Riesz basis, then this definition reduces to diagonalizability. Likewise, if the columns of F form an orthonormal (orthogonal) basis, then it is a unitary (orthogonal) matrix and this definition reduces to unitary (orthogonal) diagonalizability.

We investigate some properties of frame diagonalization, beginning with adjoints and inverses.

Theorem 3.2. *Let $A \in M_n(\mathbb{C})$ be frame diagonalizable by the frame-dual matrix pair $(F, \tilde{F})$. Then A^* is frame diagonalizable by $(\tilde{F}, F)$. Moreover, if $Av = \sum_{i=1}^m \langle v, f_i\rangle \lambda_i \tilde{f}_i$, then $A^*v = \sum_{i=1}^m \langle v, \tilde{f}_i\rangle \overline{\lambda_i} f_i$.*

Proof.

$$\langle Au, v\rangle = \sum_{i=1}^m \langle u, f_i\rangle \lambda_i \langle \tilde{f}_i, v\rangle = \langle u, \sum_{i=1}^m \overline{\lambda_i}\langle v, \tilde{f}_i\rangle f_i\rangle$$

Hence $A^*v = \sum_{i=1}^m \langle v, \tilde{f}_i\rangle \overline{\lambda_i} f_i$. □

Theorem 3.3. *Let $A = \tilde{F}\Lambda F^*$ be invertible, such that Λ is also invertible. Then there exists another dual frame matrix, $\tilde{\tilde{F}}$ such that $A^{-1} = \tilde{\tilde{F}}\Lambda^{-1}F^*$.*

In other words, for all $v \in \mathbb{C}^n$, $A^{-1}v = \sum_{i=1}^m \langle v, f_i\rangle \frac{1}{\lambda_i}\tilde{\tilde{f}}_i$.

Proof. Let $\tilde{\tilde{\mathcal{F}}}$ have columns $\tilde{\tilde{f}}_i = \lambda_i A^{-1}\tilde{f}_i$. We first show that $\{\tilde{\tilde{f}}_i\}_{i=1}^n$ is a dual frame of $\{f_i\}_{i=1}^n$.

$$\sum_{i=1}^m \langle v, f_i\rangle \tilde{\tilde{f}}_i = \sum_{i=1}^m \langle v, f_i\rangle \lambda_i A^{-1}\tilde{f}_i = A^{-1}\sum_{i=1}^m \langle v, f_i\rangle \lambda_i \tilde{f}_i = A^{-1}(Av) = v$$

$$\sum_{i=1}^{m} \langle v, \tilde{\tilde{f}}_i \rangle f_i = \sum_{i=1}^{m} \langle v, \lambda_i A^{-1} \tilde{f}_i \rangle f_i = \sum_{i=1}^{m} \overline{\lambda_i} \langle A^{-1*} v, \tilde{f}_i \rangle f_i = A^*(A^{-1*} v) = v$$

So $\tilde{\tilde{F}}$ is another dual frame matrix. For all $v \in \mathbb{C}^n$,

$$\tilde{\tilde{F}} \Lambda^{-1} F^* v = \tilde{\tilde{F}} \Lambda^{-1} (\langle v, f_i \rangle)_{i=1}^m = \tilde{\tilde{F}} (\frac{1}{\lambda_i} \langle v, f_i \rangle)_{i=1}^m = \sum_{i=1}^{m} \frac{1}{\lambda_i} \langle v, f_i \rangle \lambda_i A^{-1} \tilde{f}_i = A^{-1} v$$

So $A^{-1} = \tilde{\tilde{F}} \Lambda^{-1} F^*$ and $A^{-1} v = \sum_{i=1}^{m} \langle v, f_i \rangle \frac{1}{\lambda_i} \tilde{\tilde{f}}_i$. □

What is interesting about this theorem is that we can use it to define a relationship similar to that of eigenvectors and eigenvalues.

Corollary 3.4. *Let A be invertible such that $A = \tilde{F} \Lambda F^*$. Then there exists another dual frame $\tilde{\tilde{\mathcal{F}}}$ such that for $1 \leq i \leq m$,*

$$A \tilde{\tilde{f}}_i = \lambda_i \tilde{f}_i.$$

If Λ is invertible, such that $A^{-1} = \tilde{\tilde{F}} \Lambda^{-1} F^*$, then

$$A^{-1} \tilde{f}_i = \frac{1}{\lambda_i} \tilde{\tilde{f}}_i.$$

We remark here that given a frame $\mathcal{F}$ and its dual $\tilde{\mathcal{F}}$, there may be multiple diagonal matrices for a given matrix A. For example, let $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Then for any $x \in \mathbb{R}$,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ 0 & 4-x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ -1 & 1 \end{bmatrix}.$$

However, this is not always the case with all frames and their duals. If we change the adjoint of the dual frame matrix above to $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1 & 1 \end{bmatrix}$, then there is a unique diagonal,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1 & 1 \end{bmatrix}.$$

4. Existence of a Universal Frame Diagonalizer

When we restrict ourselves to diagonalizing with square matrices, as with S in the diagonalization $S\Lambda S^{-1}$ or U and V in the singular value decomposition $U\Lambda V^*$, we are unable to simultaneously diagonalize all matrices. However, with frame diagonalization, for any $n \in \mathbb{N}$, we can construct a frame of n^2 elements along with a dual such that any $A \in M_n(\mathbb{C})$ is frame diagonalized by this pair. We define one such universal diagonalizing frame and its dual in matrix form below.

Definition 4.1. Let $n \in \mathbb{N}$. Define $F_U(n)$ to be the $n \times n^2$ matrix consisting of n $n \times n$ block matrices $(F_U^i)_{i=1}^n$, such that the ith row of the ith block matrix F_U^i consist of ones with zeros everywhere else.

Another way to understand the block matrices $(F_U^i)_{i=1}^n$ is that every column in F_U^i is ϵ_i, where $(\epsilon_i)_{i=1}^n$ is the standard orthonormal basis in $\mathbb{C}^n$.

Definition 4.2. Let $n \in \mathbb{N}$. Define $\tilde{F}_U(n)$ to be the $n \times n^2$ matrix consisting of n $n \times n$ block matrices $(\tilde{F}_U^i)_{i=1}^n$, defined recursively as follows: let each entry of the first row and column of $\tilde{F}_U^1$ equal $1/n$, with $(n-1) \times (n-1)$ submatrix $[\tilde{F}_U^1]_{11}$ equal to a diagonal matrix with all diagonal entries equal to $-1/n$. To obtain $\tilde{F}_U^2$, interchange the first and second rows. To obtain $\tilde{F}_U^3$, interchange the second and third rows of $\tilde{F}_U^2$. Define $\tilde{F}_U^i$ recursively.

It is best to understand $F_U(n)$ and its dual visually through an example. Consider the case $n = 4$.

$$F_U(4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\tilde{F}_U^1 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \quad \tilde{F}_U^2 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\tilde{F}_U^3 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \quad \tilde{F}_U^4 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\tilde{F}_U(4) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We first verify that these frame matrices are indeed duals.

Lemma 4.3. *$\tilde{F}_U(n)$ is a dual frame matrix for $F_U(n)$.*

Proof. We verify that $F_U(n)\tilde{F}_U(n)^* = \tilde{F}_U(n)F_U(n)^* = I$. To do so, we notice that this involves the sum of products of block matrices: $F_U(n)\tilde{F}_U(n)^* = \sum_{i=1}^n F_U^i \tilde{F}_U^{i*}$, $\tilde{F}_U(n)F_U(n)^* = \sum_{i=1}^n \tilde{F}_U^i F_U^{i*}$.

Consider $F_U^i \tilde{F}_U^{i*}$. Since the ith row of F_U^i is the only nonzero row, the ith row of $F_U^i \tilde{F}_U^{i*}$ will also be the only nonzero row. Since every entry of the ith row of F_U^i is 1, the jth entry of the ith row of $F_U^i \tilde{F}_U^{i*}$ would be the sum of the entries of the jth column of $F_U(n)^*$. Notice, the (ii)-entry is 1. The rest of the entries are zero. Therefore, $\tilde{F}_U(n)F_U(n)^* = \sum_{i=1}^n \tilde{F}_U^i F_U^{i*} = I$. A similar argument gives $\tilde{F}_U(n)F_U(n)^* = I$. □

In addition to $\tilde{F}_U(n)$ being a dual of $\mathcal{F}_U(n)$, the block sub-matrices $\tilde{F}_U^i(n)$ have non-zero determinants, and are therefore invertible.

Lemma 4.4. *For $n \in \mathbb{N}$ and $1 \leq i \leq n$,*

$$\det(\tilde{F}_U^i(n)) = \begin{cases} 1/n^{n-1} & \text{if } i+n \text{ is even,} \\ -1/n^{n-1} & \text{if } i+n \text{ is odd.} \end{cases}$$

Proof. We first prove by induction that for $i = 1$, $\det(\tilde{F}_U^1(n)) = 1/n^{n-1}$ if n is odd and $-1/n^{n-1}$ if n is even. Clearly, for $n = 1$, $\det(\tilde{F}_U^1(1)) = 1$. Now suppose the statement holds true for n. To calculate $\det(\tilde{F}_U^1(n+1))$, we first calculate $\det((n+1)\cdot \tilde{F}_U^1(n+1))$. This matrix can be obtained by adding to $n \cdot \tilde{F}_U^1(n)$ a column on the end and a row along the bottom such that the $1(n+1)$-entry and the $(n+1)1$-entry are both 1, the $(n+1)(n+1)$-entry is -1, and the rest of the added entries are 0. Notice, we have

$$\det(n \cdot \tilde{F}_U^1(n)) = \begin{cases} n & \text{if } n \text{ is odd,} \\ -n & \text{if } n \text{ is even.} \end{cases}$$

To find the determinant of $(n+1)\cdot \tilde{F}_U^1(n+1)$, first, interchange the first and second columns. Then interchange the first and second rows. By interchanging twice, the determinant is unaffected. Now the matrix has -1 in its (11)-entry, 1 in its (12)-entry, and 0 for all other entries in the first row. It also has $n \cdot \tilde{F}_U^1(n)$ as its (11)-submatrix and a triangular matrix with (11)-entry equal to 1 and (ii)-entry equal to -1 for $i \neq 1$ as its (12)-submatrix.

Hence,

$$\det((n+1)\cdot \tilde{F}_U^1(n+1)) = \begin{cases} -(n+1) & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$$
$$= \begin{cases} n+1 & \text{if } n+1 \text{ is odd,} \\ -(n+1) & \text{if } n+1 \text{ is even.} \end{cases}$$

Hence,

$$\det(\tilde{F}_U^1(n+1)) = \begin{cases} 1/(n+1)^n & \text{if } n+1 \text{ is odd,} \\ -1/(n+1)^n & \text{if } n+1 \text{ is even.} \end{cases}$$

So the theorem holds for $n \in \mathbb{N}$, $i = 1$. Since $F_U^i(n)$ is obtained recursively by interchanging the $(i-1)$th and ith rows of $F_U^{i-1}(n)$, the determinants alternate between $1/n^{n-1}$ and $-1/n^{n-1}$. □

We now prove that the frame-dual matrix pair $(F_U(n), \tilde{F}_U(n))$ is a universal diagonalizer for any matrix $A \in M_n(\mathbb{C})$.

Theorem 4.5. *The frame-dual matrix pair $(F_U(n), \tilde{F}_U(n))$ frame diagonalizes any $A \in M_n(\mathbb{C})$. In other words, we can always find an $n^2 \times n^2$ diagonal matrix Λ such that $A = \tilde{F}_U(n)\Lambda F_U(n)^*$.*

Proof. We can understand $\tilde{F}_U(n)\Lambda F_U(n)^*$ in block diagonal form,

$$\begin{bmatrix} \tilde{F}_U^1(n) & \tilde{F}_U^2(n) & ... & \tilde{F}_U^n(n) \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 & ... & 0 \\ 0 & \Lambda_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \Lambda_n \end{bmatrix} \begin{bmatrix} F_U^1(n)^* \\ F_U^2(n)^* \\ ... \\ F_U^n(n)^* \end{bmatrix}$$

where for $1 \le i \le n$, Λ_i is a diagonal matrix.

This reduces to

$$\left[\tilde{F}_U^1(n)\Lambda_1 F_U^1(n)^*\right] + \left[\tilde{F}_U^2(n)\Lambda_2 F_U^2(n)^*\right] + ... + \left[\tilde{F}_U^n(n)\Lambda_n F_U^n(n)^*\right].$$

Since $F_U^i(n)^*$ is the matrix with ith column consisting of entries $1/n$ and zeros everywhere else, each of these matrices will have zeros everywhere except the ith column. Thus the ith column of this matrix is given by the ith column of $\tilde{F}_U^i(n)\Lambda_i F_U^i(n)^*$.

By Lemma 4.4, $\tilde{F}_U^i(n)$ is invertible. Hence we are able to define the vector $\begin{bmatrix} \lambda_i(1) \\ \lambda_i(2) \\ ... \\ \lambda_i(n) \end{bmatrix} = (\tilde{F}_U^i(n))^{-1} a_i$, where a_i is the ith column of A.

Let $\Lambda_i = \begin{bmatrix} \lambda_i(1) & 0 & ... & 0 \\ 0 & \lambda_i(2) & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \lambda_i(n) \end{bmatrix}$. Notice,

$$\tilde{F}_U^i(n)\Lambda_i F_U^i(n)^* = \tilde{F}_U^i(n) \begin{bmatrix} \lambda_i(1) \\ \lambda_i(2) \\ ... \\ \lambda_i(n) \end{bmatrix} = F_U^i(n)(\tilde{F}_U^i(n))^{-1} a_i = a_i,$$

therefore there exists a diagonal matrix Λ such that $A = \tilde{F}_U(n)\Lambda F_U(n)^*$. □

Example 4.6. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Although this matrix is not diagonalizable, this matrix is frame diagonalizable by the universal diagonalizer.

Using the formula $\lambda_i = (\tilde{F}_U^i(n))^{-1} a_i$, we obtain the diagonal entries of Λ:

$$\lambda_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

5. Diagonalization through Singular Value Decomposition

We showed in the previous section that we can universally frame diagonalize every matrix with a single frame-dual pair with n^2 elements. Can we do better in terms of the number of frame elements? In fact, we sometimes can. We first show in this section that we can frame diagonalize with a frame of $2n$ elements using singular value decomposition. Then in the next section, we show that we can frame diagonalize with a frame of $n+k$ elements, where $k \leq n/2$ is sufficient for diagonalization, using Jordan block decomposition.

Definition 5.1. For a given matrix $A \in M_n(\mathbb{C})$, let A have singular value decomposition $U\Lambda V^*$. Define the associated $n \times 2n$ SVD-frame matrix in block matrix form, $F(A)_{SVD} = [V\ (I - VU^*)]$ and dual frame matrix $\tilde{F}(A)_{SVD} = [U\ I]$, where I is the $n \times n$ identity matrix.

Through a quick calculation, one can determine that $\tilde{F}(A)_{SVD}$ is a dual frame matrix for $F(A)_{SVD}$. Another quick calculation will verify that this frame-dual frame matrix pair frame diagonalizes A.

Theorem 5.2. *Let $A \in M_n(\mathbb{C})$ with singular value decomposition $A = U\Lambda V^*$. Then the frame-dual matrix pair $(F(A)_{SVD}, \tilde{F}(A)_{SVD})$ diagonalizes A. In particular,*

$$A = [U\ I] \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^* \\ I - UV^* \end{bmatrix}.$$

Proof.

$$\begin{aligned} [U\ I] \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^* \\ I - UV^* \end{bmatrix} &= \begin{bmatrix} U\Lambda & 0 \end{bmatrix} \begin{bmatrix} V^* \\ I - UV^* \end{bmatrix} \\ &= [U\Lambda V^*] \\ &= A \end{aligned}$$

□

Example 5.3. Consider again the example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The beautiful singular value decomposition is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{\phi^2+1}} \begin{bmatrix} \phi & -1 \\ 1 & \phi \end{bmatrix} \begin{bmatrix} \phi & 0 \\ 0 & \frac{1}{\phi} \end{bmatrix} \frac{1}{\sqrt{\phi^2+1}} \begin{bmatrix} 1 & \phi \\ -\phi & 1 \end{bmatrix},$$

where $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio. Then by Theorem 5.2, A can be frame diagonalized in the following way:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\phi}{\sqrt{\phi^2+1}} & \frac{-1}{\sqrt{\phi^2+1}} & 1 & 0 \\ \frac{1}{\sqrt{\phi^2+1}} & \frac{\phi}{\sqrt{\phi^2+1}} & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi & 0 & 0 & 0 \\ 0 & \frac{1}{\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\phi^2+1}} & \frac{\phi}{\sqrt{\phi^2+1}} \\ \frac{-\phi}{\sqrt{\phi^2+1}} & \frac{1}{\sqrt{\phi^2+1}} \\ \frac{2-\phi}{\phi^2+1} & \frac{-\phi}{\phi^2+1} \\ \frac{\phi}{\phi^2+1} & \frac{2-\phi}{\phi^2+1} \end{bmatrix}.$$

6. Diagonalization through Jordan Block Decomposition

We have thus far shown that any general matrix in $M_n(\mathbb{C})$ can be frame diagonalized by a universal frame-dual pair with n^2 elements, and can also be frame diagonalized by using the singular value decomposition of the matrix, cutting the number of frame elements down to $2n$. We improve on this further by using the Jordan canonical form of the matrix, cutting the number of frame elements down to $\lfloor 3n/2 \rfloor$.

Recall the discussion after Proposition 2.7, in which we said that any $n \times m$ frame matrix can be completed to an $m \times m$ nonsingular matrix by adding $m - n$ pairwise linearly independent rows. Conversely, by removing $m - n$ rows of any nonsingular matrix and the corresponding $m - n$ columns of its inverse, one is left with a dual frame matrix and the adjoint of a frame matrix.

So in thinking about how to choose a frame and its dual to frame diagonalize a matrix, we can think about understanding the original $n \times n$ matrix A as a principal submatrix of a larger $m \times m$ matrix. Although this can be done in a number of ways, we focus on supmatrices which have A as their bottom right blocks.

Consider once again the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We showed earlier that this matrix can be frame diagonalized by the universal frame diagonalizer and its dual, which uses a frame with four elements. We also showed that this matrix can be frame diagonalized using the singular value decomposition, which gives us a different frame-dual matrix pair but still with four elements. Here, we show that we can get away with frame diagonalizing with a frame with three elements. We lift A to the supmatrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. The characteristic polynomial equation is $(1-x)^3 + 1 = 0$, so the three distinct eigenvalues are the third roots of unity plus 1. Let $z_n = e^{2\pi i/n}$ denote an nth root of unity. Then we have the following diagonalization.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & z_3^1 & z_3^2 \\ 1 & z_3^2 & z_3^1 \end{bmatrix} \begin{bmatrix} 1+z_3^0 & 0 & 0 \\ 0 & 1+z_3^1 & 0 \\ 0 & 0 & 1+z_3^2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & \frac{-1-\sqrt{3}i}{6} & \frac{-1+\sqrt{3}i}{6} \\ 1/3 & \frac{-1+\sqrt{3}i}{6} & \frac{-1-\sqrt{3}i}{6} \end{bmatrix}.$$

To then find the frame diagonalization for A, we cut down the eigenmatrices.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & z_3^1 & z_3^2 \\ 1 & z_3^2 & z_3^1 \end{bmatrix} \begin{bmatrix} 1+z_3^0 & 0 & 0 \\ 0 & 1+z_3^1 & 0 \\ 0 & 0 & 1+z_3^2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ \frac{-1-\sqrt{3}i}{6} & \frac{-1+\sqrt{3}i}{6} \\ \frac{-1+\sqrt{3}i}{6} & \frac{-1-\sqrt{3}i}{6} \end{bmatrix}.$$

One thing to notice about the example above, the matrix A is a Jordan block matrix. Recall the definition of a Jordan block matrix.

Definition 6.1. An $n \times n$ Jordan block matrix $J_n(\lambda)$ is a matrix of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

where there is a constant value of λ along the diagonal, ones along the superdiagonal and zeros everywhere else.

Jordan block matrices include the trivial case where $n = 1$, $J_1(\lambda) = [\lambda]$. This is the only case where the Jordan block matrix is a diagonal matrix. An important property of non-trivial Jordan block matrices is that they are not diagonalizable. This is clear to see; since $J_n(\lambda)$ is a triangular matrix, its eigenvalues are along the diagonal, thus λ is the only eigenvalue. This eigenvalue has algebraic multiplicity $n \geq 2$ but has geometric multiplicity 1. We show that although we cannot diagonalize a non-trivial Jordan block matrix, we can frame diagonalize a non-trivial Jordan block matrix by a frame-dual pair that is overcomplete by only one vector.

Theorem 6.2. *A Jordan block matrix $J_n(\lambda)$ can always be frame diagonalized by a frame-dual pair with $n+1$ elements.*

Proof. We add a row and column to the top and left sides of $J_n(\lambda)$ to form an $(n+1) \times (n+1)$ supmatrix. We first add the $1 \times m$ row $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ to form an $(n+1) \times n$ matrix, then add a $(n+1) \times 1$ column to the left,

$\begin{bmatrix} \lambda \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ to form the $(n+1) \times (n+1)$ supmatrix,

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

This matrix has characteristic polynomial equation $(\lambda - x)^{n+1} + 1 = 0$ if n is even and $(\lambda - x)^{n+1} - 1 = 0$ if n is odd. So the eigenvalues of the supmatrix are $\lambda + z_{n+1}^k$, $k = 0, 1, ..., n$. The eigenvalues are distinct since the roots of unity are distinct, so the supmatrix is diagonalizable by a nonsingular matrix S whose columns form the eigenvectors of the supmatrix. By then removing the first row of S and the first column of S^{-1}, we create the dual frame matrix $\tilde{F}$ and the adjoint of the frame matrix F^* respectively. Thus $J_n(\lambda)$ is frame diagonalizable by a frame-dual pair with $n+1$ elements. □

We can generalize this, and prove that a block diagonal matrix with k non-trivial Jordan block matrices along the diagonal can be diagonalized by a frame with $n + k$ elements. We first look at an example to get a better sense of the proof of the general case.

Notice in the proof of frame diagonalizing the Jordan block matrix, we added a λ in the upper left corner and filled in ones so that λ and 1 both appeared exactly once in each row and column. Notice, this creates a supmatrix which is a permutation matrix with an added diagonal, which we call the *perm+diag form*. We will aim to create a perm+diag form here as well, in a systematic way. We first focus on the first Jordan block, and add a row on top and column to the left so that the submatrix with the Jordan block and the appropriate section of the added row and column are in perm+diag form. We set the rest of the row and column entries to zero. We then add another row on top and column to the left so that the submatrix with the second Jordan block and the appropriate section of the newly added row and column again is in perm+diag form. By doing this to each Jordan block, we end up with an $(n+k) \times (n+k)$ matrix that is itself in perm+diag form.

Example 6.3. Consider the block diagonal matrix

$$A = \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \end{bmatrix}.$$

We first add a row on top and column to the left to make first submatrix into perm+diag form, then add another row on top and column to the left to make the second submatrix into perm+diag form.

$$\Longrightarrow \begin{bmatrix} \begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Notice, by permuting the columns $(1,2,3,4,5,6,7) \to (2,3,4,1,5,6,7)$ and then permuting the rows $(1,2,3,4,5,6,7) \to (2,3,4,1,5,6,7)$, we end up with a block diagonal matrix, such that each block is diagonalizable as seen in the proof of Theorem 6.2.

Let P be the permutation matrix $P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

$S_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & z_3^1 & z_3^2 \\ 1 & z_3^2 & z_3^1 \end{bmatrix}$, $S_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z_4^1 & z_4^2 & z_4^3 \\ 1 & z_4^2 & z_4^3 & z_4^1 \\ 1 & z_4^3 & z_4^1 & z_4^2 \end{bmatrix}$, and $S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = P \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 \end{bmatrix} P^{-1}$$

$$= PS \begin{bmatrix} 2+z_3^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+z_3^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2+z_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3+z_4^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3+z_4^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3+z_4^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3+z_4^3 \end{bmatrix} S^{-1}P^{-1}$$

Thus by removing the first two rows of PS and the first two columns of $(PS)^{-1}$, we obtain a dual and adjoint of the frame matrix respectively which diagonalize the original block matrix.

Theorem 6.4. *A block diagonal matrix $A \in M_n(\mathbb{C})$ with k non-trivial Jordan blocks down the diagonal, $J_{n_1}(\lambda_1)$, $J_{n_2}(\lambda_2)$, ..., $J_{n_k}(\lambda_k)$ where $n_1 + n_2 + ... + n_k = n$, can be frame diagonalized by a frame with $n + k$ elements.*

Proof. The proof is a generalization of the example above. Lift the original $n \times n$ matrix A to an $(n+k) \times (n+k)$ supmatrix A' with submatrix A in the bottom right corner, such that the remaining entries of A' are the following: the (i,i)-entry equals λ_i, the $(i, 1+k+\sum_{j=1}^{k-i} n_j)$-entry equals 1 (where we use the convention $\sum_{j=1}^{0} n_j = 0$), the $(k+\sum_{j=1}^{k+1-i} n_j, i)$-entry equals 1 for all $1 \leq i \leq k$, and the rest of the remaining entries are zero. We then permute the rows and columns in such a way that we obtain a block diagonal matrix, A'', with each of the original Jordan blocks augmented to perm+diag form. Since the permutations of the rows and columns were the same, we have that the row and column permutation matrices are inverses of each other. In other words, $A' = PA''P^{-1}$. Each of these blocks are diagonalizable by Theorem 6.2. Thus A'' is diagonalizable, $A'' = S\Lambda S^{-1}$. This then shows that A' is diagonalizable, $A' = (PS)\Lambda(PS)^{-1}$. Finally, by removing the first k rows of PS and the first k columns of $(PS)^{-1}$, we obtain a dual and the adjoint of a frame matrix respectively which diagonalize the original matrix A. □

Note, the eigenvalues of A do not need to be distinct, and it may be that the eigenvalues of the supmatrix are not distinct. However, because of the block diagonal structure, the permuted supmatrix will still be diagonalizable. In addition, an advantage of this particular algorithm is that unlike with the previous algorithms, this one retains information about the eigenvalues within the diagonal matrix. The nonsingular matrix S has a general form, making the frame diagonalization straightfoward.

The usefulness of the previous discussion becomes immediately evident when we consider the Jordan canonical form theorem, below.

Theorem 6.5 (Jordan canonical form theorem)**.** *Let $A \in M_n(\mathbb{C})$. Then A is similar to a Jordan matrix, i.e., there is a nonsingular matrix $S \in M_n(\mathbb{C})$ such that*

$$A = S \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2}(\lambda_2) & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & J_{n_k}(\lambda_k) & 0 \\ 0 & 0 & \cdots & 0 & \Lambda_{k+1} \end{bmatrix} S^{-1}$$

where Λ_{k+1} is a diagonal matrix (possibly 0×0), and $\lambda_1, \lambda_2, ..., \lambda_k$ along with the diagonal entries of Λ_{k+1} are the eigenvalues of A, not necessarily distinct.

This Jordan matrix is unique up to permutations of the Jordan blocks, if we return to the original definition where the diagonal entries of Λ_{k+1} are considered Jordan blocks with $n = 1$.

Thus combining Theorem 6.3 and 6.4, we obtain the following theorem.

Theorem 6.6. *Let $A \in M_n(\mathbb{C})$ have Jordan canonical form with k non-trivial Jordan blocks. Then A can be frame diagonalized by an $n \times (n + k)$ frame-dual matrix pair.*

Proof. Let J be the $n_1 \times n_1$ Jordan block diagonal form as in Theorem 6.3 with k Jordan blocks and Λ_{k+1} be the $n_2 \times n_2$ diagonal matrix, where $n_1 + n_2 = n$. Then by Theorem 6.4, A can be written $S \begin{bmatrix} J & 0 \\ 0 & \Lambda_{k+1} \end{bmatrix} S^{-1}$. By Theorem 6.3, there is a frame-dual matrix pair, $(F, \tilde{F})$ such that $J = \tilde{F}\Lambda F^*$. Then

$$A = S \begin{bmatrix} \tilde{F}\Lambda F^* & 0 \\ 0 & \Lambda_{k+1} \end{bmatrix} S^{-1} = S \begin{bmatrix} \tilde{F} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda_{k+1} \end{bmatrix} \begin{bmatrix} F^* & 0 \\ 0 & I \end{bmatrix} S^{-1}.$$

$S\begin{bmatrix}\tilde{F} & 0\\ 0 & I\end{bmatrix}$ is a dual frame matrix of the frame matrix $(S^{-1})^*\begin{bmatrix}F & 0\\ 0 & I\end{bmatrix}$:

$$\begin{aligned}\left(S\begin{bmatrix}\tilde{F} & 0\\ 0 & I\end{bmatrix}\right)\left((S^{-1})^*\begin{bmatrix}F & 0\\ 0 & I\end{bmatrix}\right)^* &= \left(S\begin{bmatrix}\tilde{F} & 0\\ 0 & I\end{bmatrix}\right)\left(\begin{bmatrix}F & 0\\ 0 & I\end{bmatrix}^* S^{-1}\right)^*\\ &= \left(S\begin{bmatrix}\tilde{F} & 0\\ 0 & I\end{bmatrix}\right)\left(\begin{bmatrix}F^* & 0\\ 0 & I\end{bmatrix} S^{-1}\right)^*\\ &= S\begin{bmatrix}\tilde{F}F^* & 0\\ 0 & I\end{bmatrix}S^{-1}\\ &= S\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}S^{-1} = SS^{-1} = I.\end{aligned}$$

The frame matrix $S\begin{bmatrix}\tilde{F} & 0\\ 0 & I\end{bmatrix}$ is an $n\times((n_1+k)+n_2) = n\times(n+k)$ matrix. □

Since the non-trivial Jordan blocks are of size 2 or greater, it must be that $k \leq \lfloor n/2 \rfloor$. Therefore, any matrix $A \in M_n(\mathbb{C})$ can be frame diagonalized by a frame-dual pair with $\lfloor 3n/2 \rfloor$ elements.

Acknowledgements

The author is grateful to Vern Paulsen for his valuable insights on the use of the singular value decomposition for frame diagonalization. The author is also grateful to Rick Denman and David Jiménez for their helpful comments and suggestions.

References

[1] P. Balazs, Basic definition and properties of Bessel multipliers, J. Math. Anal. Appl., 325(2007), no. 1, 571–585.

[2] P. Balazs, Hilbert-Schmidt Operators and Frames - Classification, Approximation by Multipliers and Algorithms, International Journal of Wavelets, Multiresolution and Information Processing, 6(2008), no. 2, 315–330.

[3] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, MA, 2003.

[4] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Regional Conference Series in Applied Math., Philadelphia, PA, 1992.

[5] H. G. Feichtinger, K. Nowak, A first survey of Gabor multipliers, in Advances in Gabor analysis, H. G. Feichtinger, T. Strohmer, eds.,Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2003, pp. 99–128.

[6] V. K. Goyal, J. Kovâcevíc, and J. A. Kelner. Quantized frame expansions with erasures, Journal of Appl. and Comput. Harmonic Analysis, 10(2001), no. 3, 203–233.

[7] C. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, MA, 2001.

[8] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, UK, 1990.