Abstract. We introduce a new definition of localization for frames which
gets rid of the dependence on the indexing of the frames. Two main results of
Gröchenig are extended to this definition, namely that the dual of a localized
frame is also localized, and a frame localized with respect to another frame
is a Banach frame for the associated family of Banach spaces. These results
parallel the results of a more recent paper by Fornasier and Gröchenig.

1. Introduction

The concept of localization of frames was recently introduced independently by
Gröchenig [?] and the group consisting of Balan, Casazza, Heil, and Landau (BCHL)
[?]. To understand localization of frames, let \( \mathcal{H} = L^2(\mathbb{R}) \) and consider two frames
\( \mathcal{F} = (f_x)_{x \in X} \) and \( \mathcal{E} = (e_y)_{y \in Y} \) of \( \mathcal{H} \) whose indexing sets \( X \) and \( Y \) are countable
subsets of \( \mathbb{R} \). We can think of each \( f_x \in \mathcal{F} \) as being “concentrated” near \( x \), and
similarly for each \( e_y \in \mathcal{E} \). Roughly, \( \mathcal{F} \) is localized with respect to \( \mathcal{E} \) if each \( f_x \) can
be well-approximated by a finite linear combination of \( e_y \)'s. In other words, since
\( f_x \) can be written as

\[
 f_x = \sum_{y \in Y} \langle f_x, e_y \rangle e_y 
\]

we say that \( \mathcal{F} \) is localized with respect to \( \mathcal{E} \) if the magnitudes of the coefficients
\( |\langle f_x, e_y \rangle| \) exhibit a certain decay as the distance between \( x \) and \( y \) increases. Equiva-

tently, \( \mathcal{F} \) is localized with respect to \( \mathcal{E} \) if the cross-Gramian matrix \( \{ \langle f_x, e_y \rangle \}_{x \in X, y \in Y} \)
has a decay off the diagonal.

 Localization has already proven to be a powerful new quality of frames [?], [?],
[?], [?], [?], [?], [?]. Gröchenig proved that a frame localized with respect
to a Riesz basis is automatically a Banach frame for an often important family
of Banach spaces associated to the Riesz basis. This result generalized results
from sampling theory, time-frequency analysis, and wavelet analysis [?], [?], [?], [?].
Later, Fornasier and Gröchenig [?] proved a similar result for frames localized with
respect to other frames, where the localization is defined by the Gramian matrices
belonging to a particular spectral algebra. The role of Banach matrix algebras in
this theory of localized frames was further developed by Gröchenig and Leinert
[?]. Most recently, Fornasier and Rauhut [?] introduced the notion of continuous
localized frames indexed by a locally compact space endowed with a Radon measure
and showed that these frames can be sampled to create discrete localized frames.
Independent of these developments, BCHL introduced the notion of localized frames
to prove powerful results concerning the density and excess of frames, extending
their results in [?] and [?]. They also provided an illuminating new perspective on
previously known results concerning Gabor frames.
In this paper, we introduce a definition which gets rid of the dependence of localization on the indexing of the frames. The property of localization was dependent on indexing for all previous definitions. We then show that the main results of Gröchenig still hold. This definition also extends the results of BCHL; these results can be found in [7]. It must be remarked that this definition was developed prior to knowledge of the following papers, [8], [9], [10]. The Fornasier and Rauhut definition of localization [10] may be thought of as the most general definition, as this definition is for continuous frames indexed on a locally compact space with Radon measure. However, the localization condition depends on the prescribed choice of Radon measure, which corresponds to a particular choice of arrangement of the index set. Additionally, in the papers just mentioned, the localization condition is defined for continuous frames indexed on a locally compact space with Radon measure. We expect that this definition can be extended using more general Banach algebras.

Our paper will be organized as follows. In Section 2, we introduce the symmetric definition of localization. In Section 3, we investigate the equivalence structure of \( \ell^1 \)-self-localized frames, and finally, in Section 4, we extend the two main results of Gröchenig, namely localization of the dual frame, and the construction of Banach frames.

2. Symmetric Localization

Before introducing the new definition, we fix basic notation. We recommend [1], [8], [9], [10], and [11] for additional background.

For a countable set \( X \), let \( \mathcal{F} = (f_x)_{x \in X} \) be a frame for a separable Hilbert space \( \mathcal{H} \) with frame bounds \( A, B \). The analysis operator will be denoted \( C := C_{\mathcal{F}} : \mathcal{H} \to \ell^2(X) \), where \( C(f) = (\langle f, f_x \rangle)_{x \in X} \). The synthesis operator will be denoted \( D := D_{\mathcal{F}} : \ell^2(X) \to \mathcal{H} \), where \( D((c_x)_{x \in X}) = \sum_{x \in X} c_x f_x \). \( D \) is the adjoint of \( C \), \( D = C^* \). The frame operator denoted \( S = DC : \mathcal{H} \to \mathcal{H} \), \( S f = \sum_{x \in X} \langle f, f_x \rangle f_x \) is a positive, invertible operator such that \( A \cdot I \leq S \leq B \cdot I \).

The canonical dual frame of \( \mathcal{F} \) is denoted \( \hat{\mathcal{F}} = (S^{-1} f_x)_{x \in X} = (\hat{f}_x)_{x \in X} \) and is such that \( f = \sum_{x \in X} \langle f, \hat{f}_x \rangle f_x = \sum_{x \in X} \langle f_x, \hat{f} \rangle f_x \) for all \( f \in \mathcal{H} \).

In the following, let \( G \) be a group of the form \( \prod_{i=1}^{d} \mathbb{Z} \times \prod_{j=1}^{e} \mathbb{Z}_{b_j} \). For every \( g = (a_1 n_1, a_2 n_2, ..., a_d n_d, m_1, m_2, ..., m_e) \in G \), let

\[
|g| = \sup \{ |a_1 n_1|, |a_2 n_2|, ..., |a_d n_d|, \delta(m_1), \delta(m_2), ..., \delta(m_e) \}
\]

where \( \delta(m) = \begin{cases} 
0 & \text{if } m=0; \\
1 & \text{otherwise.} 
\end{cases} \)

Define a metric on \( G \) to be \( d(g, h) = |g - h| \) for \( g, h \in G \).

Let \( S_n(j) \) be defined to be the ball of radius \( n \) centered at \( j \) in \( G \). We define \( |S_n(j)| := \# |S_n(j)| \), the cardinality of \( S_n(j) \).

**Definition 2.1 (Symmetric localization).** Let sequences \( \mathcal{F} = (f_x)_{x \in X} \) and \( \mathcal{E} = (e_y)_{y \in Y} \) in a Hilbert space \( \mathcal{H} \), \( X \) and \( Y \) arbitrary index sets.
(1) \((F, E)\) is symmetrically \(\ell^p\)-localized if there exist maps \(a_X : X \rightarrow G\), \(a_Y : Y \rightarrow G\) such that \(\max \{\sup_{j \in \mathbb{G}} |a_X^{-1}(j)|, \sup_{j \in \mathbb{G}} |a_Y^{-1}(j)|\} \leq K < \infty\), and \(r \in \ell^p(G)\) such that for all \(x \in X, y \in Y\),

\[
|\langle f_x, e_y \rangle| \leq r_{a_X(x) - a_Y(y)}.
\]

(2) \(F\) is symmetrically \(\ell^p\)-self-localized if it is symmetrically \(\ell^p\)-localized with respect to itself.

(3) \((F, E)\) has uniform \(\ell^p\) column decay if for every \(\epsilon > 0\) there is a \(N_\epsilon > 0\) such that for all \(y \in Y\),

\[
\sum_{x \in X \setminus a_X^{-1}(\mathbb{N}_\epsilon(a_Y(y)))} |\langle f_x, e_y \rangle|^p < \epsilon.
\]

(4) \((F, E)\) has uniform \(\ell^p\) row decay if for every \(\epsilon > 0\) there is a \(N_\epsilon > 0\) such that for all \(x \in X\),

\[
\sum_{y \in Y \setminus a_Y^{-1}(\mathbb{N}_\epsilon(a_X(x)))} |\langle f_x, e_y \rangle|^p < \epsilon.
\]

**Remark 2.2.** The terms column and row decay come from considering the cross-Gramian matrix \((\langle f_x, e_y \rangle)_{x \in X, y \in Y}\).

**Remark 2.3.** If we let \(Y = G\) and \(a_Y = \text{id}\), then we have the definition of BCHL provided \(\sup_{j \in \mathbb{G}} |a_X^{-1}(j)| \leq K < \infty\). Bounded point inverses is not only desired in applications but also used in nearly all of the theorems of BCHL so it is not a restrictive condition.

Though we do not have a straight generalization of Gröchenig’s definition, every frame localized with respect to a Riesz basis in Gröchenig’s sense is localized in this symmetric sense as will be made clear below.

**Definition 2.4.** A set \(X \subseteq \mathbb{R}^d\) is separated if there exists a positive constant \(c\) such that for every \(x, y \in X\) such that \(x \neq y\), \(0 < c \leq |x - y|\).

**Definition 2.5 (Gröchenig [2]).** The frame \(F = (f_x)_{x \in X}\) in \(L^2(\mathbb{R}^d)\) is polynomially localized with respect to the Riesz basis \(E = (e_y)_{y \in Y}\) in \(L^2(\mathbb{R}^d)\), with decay \(s > 0\) (or \(s\)-localized), where \(X\) is a finite union of separated sets of \(\mathbb{R}^d\) and \(Y\) is a separated set of \(\mathbb{R}^d\), if for all \(x \in X, y \in Y, C > 0\),

\[
|\langle f_x, e_y \rangle| \leq C(1 + |x - y|)^{-s} \quad \text{and} \quad |\langle f_x, \bar{e}_y \rangle| \leq C(1 + |x - y|)^{-s}.
\]

Likewise, \(F\) is called exponentially localized with exponent \(\alpha > 0\) if for some \(\alpha > 0\) and \(C > 0\),

\[
|\langle f_x, e_y \rangle| \leq C e^{-\alpha|x-y|} \quad \text{and} \quad |\langle f_x, \bar{e}_y \rangle| \leq C e^{-\alpha|x-y|}.
\]

To prove that every polynomially localized frame is symmetrically localized, let \(F = (f_x)_{x \in X}\) be \(s\)-localized with respect to a Riesz basis \(E = (e_y)_{y \in Y}\) as above. Let \(G := \frac{1}{2\pi} \mathbb{Z}^d\). For \(x = (x_i)_{i=1}^d \in \mathbb{R}^d\), let \(n_i \leq x_i < n_i + \frac{1}{2\pi}\) with \(n_i \in \frac{1}{2\pi}\mathbb{Z}, i = 1,...d\).

We define \(a_X : X \rightarrow G\) in the following way:

\[
a_X(x) = (w_1, ..., w_d) \quad \text{where} \quad w_i = \begin{cases} n_i & \text{if } x_i < n_i + \frac{1}{2\pi} \\ n_i + \frac{1}{2\pi} & \text{if } x_i \geq n_i + \frac{1}{2\pi} \end{cases}
\]

We define \(a_Y : Y \rightarrow G\) similarly.
If we define \( r : G \to \mathbb{C} \) to be \( r = (r_g)_{g \in G} = (\mathcal{C}(\frac{1}{2}|g|^{-s}))_{g \in G} \), then \( r \in \ell^p \) where \( p > \frac{1}{2} \), and \( \mathcal{F} \) is symmetrically \( \ell^p \)-localized with respect to \( \mathcal{E} \) with decay given by \( r \). A similar argument holds for exponentially localized frames.

The most compelling reason for the introduction of this new definition for localized frames is the need for a definition that not only compare any two frames, but two frames regardless of the indexing. Recall that the frame series converges unconditionally, or equivalently, the convergence of the partial sums is independent of the indexing of the frames. All previous definitions depend on the indexing. Given a frame \( \mathcal{F} \) localized with respect to a Riesz basis \( \mathcal{E} \) in the sense of Gröchenig \([?]\), we can easily permute the index set of \( \mathcal{F} \), \( \mathcal{E} \), or both so that \( \mathcal{F} \) is no longer localized with respect to \( \mathcal{E} \). The same is true for Fornasier and Gröchenig’s definition \([?]\), Fornasier and Rauhut’s definition \([?]\), and Gröchenig and Leinert’s definition \([?]\).

The definition found in \([?]\) is also dependent on the indexing of the frames because of the dependence on the map \( a_X : X \to G \). The symmetrically localized definition is independent of the indexing. For this definition, we need only the existence of some maps \( a_X : X \to G \) and \( a_Y : Y \to G \) with finite point inverses, so given a permutation \( p \) of the index set \( X \), we can define a new map \( a'_X = a_X \circ p^{-1} : p(X) \to G \).

**Example 2.6.** Let \( \mathcal{F} = \left( \frac{\sin(\pi(x+k))}{\pi(x+k)} \right)_{k \in \frac{1}{2}\mathbb{Z}} \) and \( \mathcal{E} = \left( \frac{\sin(\pi(x+n))}{\pi(x+n)} \right)_{n \in \mathbb{Z}} \) be frames for \( L^2(\mathbb{R}) \). Let \( a_{\frac{1}{2}\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z} \) be the identity function, and \( a_{\frac{1}{2}\mathbb{Z}} : \frac{1}{2}\mathbb{Z} \to \mathbb{Z} \) be defined

\[
a_{\frac{1}{2}\mathbb{Z}}(k) = \begin{cases} 
  k & \text{if } k \in \mathbb{Z}; \\
  k - 1/2 & \text{if } k \in \mathbb{Z}^+; \\
  k + 1/2 & \text{if } k \in \mathbb{Z}^-.
\end{cases}
\]

\( \mathcal{F} \) is symmetrically \( \ell^p \) localized with respect to \( \mathcal{E} \) for any \( p > 1 \), where the decay is given by \( r_g = \begin{cases} 
  1 & \text{if } g = 0; \\
  \frac{1}{|g|^\pi} & \text{if } g \neq 0.
\end{cases} \)

**Example 2.7.** Let \( \mathcal{G}(\gamma, \Gamma) = \{ T_x M_w \gamma : (x, w) \in \Gamma \} \) and \( \mathcal{G}(\gamma, \Lambda) = \{ T_y M_z \gamma : (z, y) \in \Lambda \} \) be Gabor frames in \( L^2(\mathbb{R}) \), where \( T_x \) is the translation operator defined \( T_x f(y) = f(y-x) \) and \( M_z \) is the modulation operator defined \( M_z f(y) = e^{2\pi izy} f(y) \). Suppose \( \gamma = e^{-\pi x^2} \), the normalized Gaussian, and \( \Gamma, \Lambda \subset \mathbb{R}^2 \) with finite multiplicities. Let \( a_{\Gamma} : \Gamma \to \mathbb{Z} \times \mathbb{Z} \) and \( a_{\Lambda} : \Lambda \to \mathbb{Z} \times \mathbb{Z} \) be functions sending a point to the nearest lattice point. More concretely, suppose for \( m, n \in \mathbb{Z} \) and \( 0 < a, b < 1 \),

\[
a_{\Gamma}(x, w) = a_{\Gamma}(m + a, n + b) = \begin{cases} 
  (m, n) & \text{if } a < 1/2, b < 1/2; \\
  (m, n+1) & \text{if } a < 1/2, b \geq 1/2; \\
  (m+1, n) & \text{if } a \geq 1/2, b < 1/2; \\
  (m+1, n+1) & \text{if } a \geq 1/2, b \geq 1/2.
\end{cases}
\]

Then \( \mathcal{G}(\gamma, \Gamma) = \{ T_x M_w \gamma : (x, w) \in \Gamma \} \) is symmetrically \( \ell^1 \)-localized with respect to \( \mathcal{G}(\gamma, \Lambda) = \{ T_y M_z \gamma : (z, y) \in \Lambda \} \), where

\[
r = (r_{(m,n)})_{\mathbb{Z} \times \mathbb{Z}} = (2^{-1/2} e^{-\pi((m-1)/2 + (n-1)/2)^2})_{\mathbb{Z} \times \mathbb{Z}}.
\]

3. **Equivalence Class Structure**

The symmetries in this definition allow for a natural equivalence class structure, as was shown for Fornasier’s intrinsically localized frames in \([?]\).
Definition 3.1. Let \( S^1 := \{ F = (f_x)_{x \in X} \mid F \) is a symmetrically \( \ell^1 \)-self-localized frame of \( \mathcal{H} \}\). For \( F, E \in S^1 \), we define the relation \( F \sim E \) if \( F \) is symmetrically \( \ell^1 \)-localized with respect to \( E \).

We have the following theorem.

Theorem 3.2. \( [?] \) Let \( F \in S^1 \). Then for \( \tilde{F}, \tilde{F} \in S^1 \) and \( F \sim \tilde{F} \).

This theorem applies directly as our definition coincides with that of BCHL in the case of self-localization.

Before verifying that we have an equivalence relation, let us first prove the following proposition.

Proposition 3.3. Let \( F = (f_x)_{x \in X} \) and \( E = (e_y)_{y \in Y} \) be frame sequences for Hilbert space \( \mathcal{H} \), \( X \) and \( Y \) arbitrary index sets. Let \( a_X : X \to G \), \( a_Y : Y \to G \) be associated maps. Suppose the following conditions are satisfied:

1. \( F \) is symmetrically \( \ell^1 \)-localized with respect to \( E \), i.e., there exists \( r \in \ell^1(G) \) such that \( \| \langle f_x, e_y \rangle \| \leq r_{a_X(x) - a_Y(y)} \).

2. \( F \) is symmetrically \( \ell^1 \)-localized with respect to \( \tilde{E} \), i.e., there exists \( s \in \ell^1(G) \) such that \( \| \langle f_x, \tilde{e}_y \rangle \| \leq s_{a_X(x) - a_Y(y)} \).

Then \( F \in S^1 \) and \( E \in S^1 \).

Proof. We define convolution for \( \ell^p(G) \) in the following way:

\[
(c_j)_{j \in G} = (b_j)_{j \in G} * (d_j)_{j \in G} = \left( \sum_{k \in G} b_k d_{j-k} \right)_{j \in G}.
\]

\[
\| (f_x, f_z) \| = \left( \sum_{y \in Y} \| \langle f_x, e_y \rangle \| \right)^2 \| \langle \tilde{e}_y, f_z \rangle \|
\leq \sum_{y \in Y} \| \langle f_x, e_y \rangle \| \| \langle \tilde{e}_y, f_z \rangle \|
\leq \sum_{y \in Y} r_{a_X(x) - a_Y(y)} s_{a_Y(y) - a_X(z)}
= \sum_{j \in G} \sum_{y \in a_Y^{-1}(j)} r_{a_X(x) - a_Y(y)} s_{a_Y(y) - a_X(z)}
\leq \sum_{j \in G} K r_{a_X(x) - j} s_{j - a_X(z)}
= K (r * s)_{a_X(x) - a_X(z)}.
\]

As \( r, s \in \ell^1(G) \), we have that \( r * s \in \ell^1(G) \). Hence \( F \in S^1 \). By Theorem 3.2, \( \tilde{F} \in S^1 \), so there is a \( q \in \ell^1(G) \) such that \( \| (\tilde{f}_z, \tilde{f}_z) \| \leq q_{a_X(x) - a_X(z)} \). Then by a similar calculation as above,

\[
\| (e_y, \tilde{f}_z) \| \leq \sum_{x \in X} \| (e_y, f_z) \| \| (\tilde{f}_z, \tilde{f}_z) \| \leq K (r * q)_{a_Y(y) - a_X(z)}.
\]

Finally,

\[
\| (e_y, e_z) \| \leq \sum_{x \in X} \| (e_y, \tilde{f}_z) \| \| (f_x, e_z) \| \leq K (r * q * r)_{a_Y(y) - a_Y(z)}.
\]
As \( r, q \in \ell^1(G) \), \( r \ast q \ast r \in \ell^1(G) \). Hence \( \mathcal{E} \in S^1 \). \qed

**Proposition 3.4.** Let \( \mathcal{F} = (f_x)_{x \in X} \) and \( \mathcal{E} = (e_y)_{y \in Y} \) be frame sequences for Hilbert space \( \mathcal{H} \), \( X \) and \( Y \) arbitrary index sets. Let \( a_X : X \to G \), \( a_Y : Y \to G \) be associated maps. Suppose the following are satisfied,

1. \( \mathcal{E} \in S^1 \),
2. \( \mathcal{F} \) is symmetrically \( \ell^1 \)-localized with respect to \( \mathcal{E} \), i.e., there exists \( r \in \ell^1(G) \) such that \( |\langle f_x, e_y \rangle| \leq r_{a_X(x)-a_Y(y)} \).

Then \( \mathcal{F} \) is \( \ell^1 \) localized with respect to \( \tilde{\mathcal{E}} \), and \( \mathcal{F} \in S^1 \).

**Proof.** By theorem 3.2, if \( \mathcal{E} \in S^1 \), then \( \tilde{\mathcal{E}} \in S^1 \). So let \( s \in \ell^1(G) \) such that \( |\langle \tilde{e}_y, \tilde{e}_z \rangle| \leq s_{a_Y(y)-a_Y(z)} \). If \( K = \sup_{j \in G} |a_Y^{-1}(j)| \), then

\[
|\langle f_x, \tilde{e}_y \rangle| \leq \sum_{y \in Y} |\langle f_x, e_z \rangle| |\langle \tilde{e}_z, \tilde{e}_y \rangle| \\
\leq \sum_{y \in Y} r_{a_X(x)-a_Y(y)} s_{a_Y(y)-a_Y(y)} \\
= \sum_{j \in G} \sum_{y \in a_Y^{-1}(j)} r_{a_X(x)-a_Y(y)} s_{a_Y(y)-a_Y(y)} \\
\leq \sum_{j \in G} K r_{a_X(x)-a_Y(y)} s_{j-a_Y(y)} \\
= K(r \ast s)_{a_X(x)-a_Y(y)}.
\]

As \( r, s \in \ell^1(G) \), \( r \ast s \in \ell^1(G) \). Hence \( \mathcal{F} \) is \( \ell^1 \) localized with respect to \( \tilde{\mathcal{E}} \). So by Proposition 3.3, \( \mathcal{F} \in S^1 \). \qed

**Theorem 3.5.** \( \sim \) is an equivalence relation on \( S^1 \).

**Proof.** Reflexivity: By definition, \( \mathcal{F} \sim \mathcal{F} \).

Symmetry: It is clear to see that \( \mathcal{F} \sim \mathcal{E} \Rightarrow \mathcal{E} \sim \mathcal{F} \).

Transitivity: Let \( \mathcal{F} = (f_x)_{x \in X} \), \( \mathcal{E} = (e_y)_{y \in Y} \), \( \mathcal{G} = (g_z)_{z \in Z} \in S^1 \), such that \( \mathcal{F} \sim \mathcal{E} \) and \( \mathcal{E} \sim \mathcal{G} \). Let \( K = \sup_{j \in G} |a_Y^{-1}(j)| \). By Proposition 3.4 and symmetry, we have that \( \tilde{\mathcal{E}} \sim \tilde{\mathcal{G}} \). Let \( r \in \ell^1(G) \) and \( s \in \ell^1(G) \) be such that \( |\langle f_x, e_y \rangle| \leq r_{a_X(x)-a_Y(y)} \) and \( |\langle \tilde{e}_y, \tilde{e}_z \rangle| \leq s_{a_Y(y)-a_Z(z)} \). Then we have

\[
|\langle f_x, g_z \rangle| \leq \sum_{y \in Y} |\langle f_x, e_y \rangle| |\langle \tilde{e}_y, g_z \rangle| \\
\leq \sum_{y \in Y} r_{a_X(x)-a_Y(y)} s_{a_Y(y)-a_Z(z)} \\
= \sum_{j \in G} \sum_{y \in a_Y^{-1}(j)} r_{a_X(x)-a_Y(y)} s_{a_Y(y)-a_Z(z)} \\
= \sum_{j \in G} K r_{a_X(x)-a_Y(y)} s_{j-a_Z(z)} \\
\leq K(r \ast s)_{a_X(x)-a_Z(z)}.
\]

Notice, \( r, s \in \ell^1(G) \), so \( r \ast s \in \ell^1(G) \). Hence \( \mathcal{F} \sim \mathcal{G} \). \qed
From this equivalence class structure, we obtain the following.

**Corollary 3.6.** For all $\mathcal{F}, \mathcal{E} \in S^1$, $\mathcal{F} \sim \mathcal{E}$ implies $\mathcal{F} \sim \tilde{\mathcal{E}}, \tilde{\mathcal{F}} \sim \mathcal{E}, \tilde{\mathcal{F}} \sim \tilde{\mathcal{E}}$.

**Example 3.7.** Gabor frames for $L^2(\mathbb{R}^d)$ are modulations and translations of a single function, called an atom. These atoms ought to have good decay in both time and frequency, and a class of functions with such a property is the modulation space, $M^1$. $M^1$ consists of all functions $f$ such that the short-time Fourier transform $V_g f : \mathbb{R}^{2d} \to \mathbb{C}$ defined $V_g f(x, \omega) = \langle f, M_{\omega} T_x g \rangle$ is in $L^1(\mathbb{R}^{2d})$. In [?], we have the following theorem.

**Theorem 3.8.**[?] Let $G(\gamma, \Gamma) = (T_x M_w \gamma : (x, w) \in \Gamma)$ and $G(\lambda, G) = (T_y M_z \lambda : (z, y) \in G)$ be Gabor frames in $L^2(\mathbb{R})$, where $\gamma, \lambda \in M^1$ and $G = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$. Then the following statements hold:

(a) $G(\gamma, \Gamma), G(\lambda, G) \in S^1$,

(b) $G(\gamma, \Gamma) \sim G(\lambda, G)$.

Using this theorem and corollary 3.6, we have that all Gabor frames with generators in the modulation space $M^1$ and their canonical duals, regardless of whether or not their indices have a lattice structure, are in the same equivalence class.

As previously mentioned, a similar relation was given in [?] and [?], which was brought to the attention of the author after this paper was nearly completed. Their relation was almost an equivalence relation, and was defined on the set of frames whose Gramian matrices lie in a solid, inverse closed, involutive Banach algebra. In contrast, the set of Gramian matrices of $\ell^1$-self localized frames forms an algebra, but not necessarily an inverse closed Banach algebra. However, a frame $\mathcal{F}$ is $\ell^1$-self localized if and only if a frame $\mathcal{F}'$ is localized in the sense of Fornasier and Gröchenig, i.e., the Gramian matrix is in the Sjöstrand algebra (see Theorem A.1, Remark A.2, and Lemma A.1 in [?]).

4. Extending the Results of Gröchenig

Gröchenig had two main results in [?], that a frame is localized with respect to a Riesz basis if and only if its canonical dual exhibits the same localization property, and frames localized with respect to a Riesz basis are automatically Banach frames for the family of Banach spaces naturally associated to the Riesz basis. We shall extend both of his results as a consequence of the equivalence relation.

4.1. Localization of the Dual. In Gröchenig’s definition, we have that a frame $\mathcal{F}$ is $s$-localized with respect to a Riesz basis $\mathcal{E}$ if we have polynomial decay with respect to both $\mathcal{E}$ and its canonical dual, $\tilde{\mathcal{E}}$. Since we have that $s$-localization implies symmetric $\ell^p$-localization with $p > \frac{1}{s}$, assume $s > 1$. Then we have that $\mathcal{F}$ is symmetrically $\ell^1$-localized with respect to $\mathcal{E}$. By Proposition 3.3, if $\mathcal{F} = (f_x)_{x \in X}$ is $s$-localized with respect to a Riesz basis $\mathcal{E} = (e_y)_{y \in Y}$, $s > 1$, we have $\mathcal{F} \in S^1$ and $\mathcal{E} \in S^1$. So by Corollary 3.6, if $\mathcal{F} = (f_x)_{x \in X}$ is $s$-localized with respect to a Riesz basis $\mathcal{E} = (e_y)_{y \in Y}$, $s > 1$, then as $\mathcal{F} \sim \mathcal{E}$, we have $\mathcal{F} \sim \mathcal{E}$.

4.2. Construction of Banach Frames. One of the main goals regarding Banach frames is their construction. In particular, we would like to show that a self-localized frame defines a family of Banach spaces, and any other frame localized with respect to it is also a Banach frame for these spaces. As an example, consider...
their significance in the study of Gabor frames. Here, the Banach spaces $H^p_m$ associated with a Gabor frame are all important modulation spaces \cite{[1]}. We shall prove that if $\mathcal{F} = \{f_x\}_{x \in \mathcal{X}}, \mathcal{E} = \{(y)\}_{y \in \mathcal{Y} \in S^1}$, and $\mathcal{F} \sim \mathcal{E}$, then $\mathcal{F}$ is a Banach frame for the natural family of Banach spaces associated to $\mathcal{E}$. Note that when dealing with weights, we need extra conditions on the indices for everything to make sense.

**Definition 4.1.** Let $(B, \| \cdot \|_B)$ be a Banach space and let $(B_d(X), \| \cdot \|_{B_d})$ be a Banach space of sequences indexed by $X$. A (countable) subset $(f_x : x \in X)$ of $B'$, the dual of $B$, is called a **Banach frame** for $B$ if the following properties hold:

(a) The coefficient operator $C_{\mathcal{E}} : B \to B_d(X)$ defined by $C_{\mathcal{E}} f = (f_x(f))_{x \in X}$ is bounded.

(b) We have the norm equivalence $\|f\|_B \simeq \|f_x(f)\|_{B_d}$.

(c) There exist a bounded operator $R : B_d(X) \to B$, called the reconstruction operator, such that $R((f_x(f))_{x \in X}) = f$.

**Definition 4.2.** For $1 \leq p < \infty$, the **weighted $\ell^p$-space** $\ell^p_m(Y)$ on the index set $Y \subset \mathbb{R}^d$ is defined by the norm

$$\|c\|_{\ell^p_m} = \left( \sum_{y \in Y} |c_y|^p m(y)^p \right)^{1/p}$$

with the usual modification for $p = \infty$. The weight $m$ is a non-negative function on $\mathbb{R}^d$ which we may assume without loss of generality is continuous.

For the purposes of this paper, we assume that the weight is submultiplicative, i.e., $m(j + k) \leq m(j)m(k)$ for all $j, k \in \mathbb{R}^d$.

**Definition 4.3.** Let $\mathcal{E} = (e_y)_{y \in \mathcal{Y}}$ be a frame for $\mathcal{H}$ such that $\mathcal{E} \subset S^1$ and $\tilde{\mathcal{E}}$ be the canonical dual frame. Let $\mathcal{H}_0 \subset \mathcal{H}$ be the subspace of finite linear combinations of elements in $\mathcal{E}$. For $0 < p < \infty$ and $m$ a weight function, we define a (quasi-) norm on $\mathcal{H}_0$ by

$$\|f\|_{\mathcal{H}^p_m} = \|(f, \tilde{e}_y)_{y \in \mathcal{Y}}\|_{\ell^p_m}.$$ 

For $1 \leq p < \infty$, the **associated space** $\mathcal{H}^p_m(\mathcal{E}, \tilde{\mathcal{E}})$ is defined to be the norm completion of $\mathcal{H}_0$ with the norm $\| \cdot \|_{\mathcal{H}^p_m}$. For $p = \infty$, $\mathcal{H}^\infty_m(\mathcal{E}, \tilde{\mathcal{E}})$ is defined to be the completion of $\mathcal{H}_0$ in the $\sigma(\mathcal{H}, \mathcal{H}_0)$-topology.

$\mathcal{H}^p_m(\mathcal{E}, \tilde{\mathcal{E}})$ is a Banach space for $1 \leq p < \infty$ with $\|f\|_{\mathcal{H}^\infty_m} = \inf\{\|c\|_{\ell^\infty_m} : c \in \ell^\infty_m, f = \sum_{y \in \mathcal{Y}} c_y e_y \}$ as proved in \cite{[2]}, using the following lemma.

**Lemma 4.4.** Let $\mathcal{F}$ be symmetrically $\ell^1$-localized with respect to $\mathcal{E}$ and consider the cross-Gramian matrix $A = \{(c_y, f_x)\}_{y \in \mathcal{Y}, x \in \mathcal{X}}$. Let $c$ be a finite sequence, then $A$ acts on $c$ in the following way: $(Ac)_{x \in \mathcal{X}} = (\sum_{y \in \mathcal{Y}} (c_y, f_x)c_y)_{x \in \mathcal{X}}$. Then $A$ extends to a bounded operator from $\ell^p_m(\mathcal{Y})$ to $\ell^p_m(\mathcal{X})$, where $m$ is a submultiplicative weight. If $m = 1$, then $X, Y$ can be arbitrary countable indices. If $m \neq 1$, then we assume $X, Y, G \subset \mathbb{R}^d$, $m : \mathbb{R}^d \to \mathbb{R}$ and the maps $a_x : X \to G, a_y : Y \to G$ coming from the localization are such that $\max\{|x - a_X(x)|, |y - a_Y(y)|\} \leq \mu$ for some $\mu > 0$. 

Proof. Let \( c \in \ell^p_m(Y) \). Define \( d = (d_j)_{j \in G} \), where \( d_j = \sum_{y \in a^{-1}_v(j)} |c_y| \). Then \( d \in \ell^p_m(G) \) as \( |a^{-1}_v(j)| \leq K \) for all \( j \in G \). Hence,

\[
\|(Ac)_x\| = \left| \sum_{y \in Y} (e_y, f_x)c_y \right| \\
\leq \sum_{y \in Y} \|e_y, f_x\| |c_y| \\
\leq \sum_{j \in G} \sum_{y \in a^{-1}_v(j)} r_{a_X(x)-a_Y(y)} |c_y| \\
= \sum_{j \in G} r_{a_X(x)-j} d_j \\
= (r * d)_{a_X(x)}.
\]

Hence, by a proof found in [?], there exists some constant \( C \) such that

\[
\|Ac\|_{\ell^p_m(X)} \leq \|r * d\|_{\ell^p_m(G)} \leq C \|r\|_{\ell^p_m(G)} \|d\|_{\ell^p_m(G)} < \infty.
\]

We now prove that \( \|d\|_{\ell^p_m(G)} \leq MK\|c\|_{\ell^p_m(Y)} \), where \( M = \sup_{|z| \leq m} m(z) \) and \( K = \max\{\sup_{j \in G} a^{-1}_X(j), \sup_{j \in G} a^{-1}_Y(j)\} \). Notice, as \( m \) is assumed to be continuous, we have that \( M \) is finite.

For \( 1 \leq p < \infty \), since \( \left( \sum_{y \in a^{-1}_v(j)} |c_y| \right)^p \leq K^p \sum_{y \in a^{-1}_v(j)} |c_y|^p \) and \( m \) is submultiplicative, we have

\[
\|d\|^p_{\ell^p_m(G)} = \sum_{j \in G} d_j^pm(j)^p \\
\leq K^p \sum_{j \in G} \sum_{y \in a^{-1}_v(j)} |c_y|^p m(j)^p \\
\leq K^p \sum_{j \in G} \sum_{y \in a^{-1}_v(j)} |c_y|^p m(y)^p m(j-y)^p \\
\leq M^p K^p \sum_{j \in G} \sum_{y \in a^{-1}_v(j)} |c_y|^p m(y)^p \\
= M^p K^p \|c\|_{\ell^p_m(Y)}^p.
\]

For \( p = \infty \),

\[
\|d\|_{\ell^\infty_m(G)} = \esssup_{j \in G} d_j m(j) \\
= \esssup_{j \in G} \sum_{y \in a^{-1}_v(j)} |c_y| m(j) \\
\leq \esssup_{j \in G} \sum_{y \in a^{-1}_v(j)} |c_y| m(y) m(j-y) \\
\leq \esssup_{y \in Y} \sum_{y \in a^{-1}_v(j)} |c_y| m(y) M \\
\leq MK \|c\|_{\ell^\infty_m(Y)}.
\]

Hence, \( \|Ac\|_{\ell^p_m(X)} \leq M K C \|r\|_{\ell^p_m(G)} \|c\|_{\ell^p_m(Y)} \).
As noted in [7], the elements of $H_m^p$ are technically equivalence classes of Cauchy sequences of elements of $H_0$. Though $H_m^p$ may happen to be $\{0\}$ or dependent on the choice of duals, if the frame is $\ell^1$-self-localized, then $H_m^p$ is a Banach space independent of the choice of a dual and all self-localized frames in a particular equivalence class will be Banach frames for the same space.

**Theorem 4.5.** Suppose $F = (f_x)_{x \in X}$ and $E = (e_y)_{y \in Y}$ are frames for a Hilbert space $H$, $F, E \subseteq S^1$. If $F \sim E$, then $F$ is a Banach frame for the family of Banach spaces $H^p(E, E)$.

**Proof.** We will need to satisfy the following conditions:

(a) The coefficient operator $C : H^p \rightarrow \ell^p(X)$ defined $Cf = \langle \langle f, f_x \rangle \rangle_{x \in X}$ is bounded.

(b) There exists a bounded operator $R$ from $\ell^p(X)$ to $H^p$, called the reconstruction operator, such that $R(\langle \langle f, f_x \rangle \rangle_{x \in X}) = f$.

(c) We have the norm equivalence $\|f\|_{H^p} \approx \|\langle f, f_x \rangle\|_{\ell^p(X)}$.

Let $B = H^p$ and $B_d = \ell^p(X)$, where $1 \leq p \leq \infty$. If $f \in H^p \subseteq H$, then $f = \sum_{y \in Y} c_y e_y$ where $c \in \ell^p(Y)$. We define the linear functionals $(f_x)_{x \in X} \in (H^p)'$ in the following way:

$$f_x(f) = \langle f, f_x \rangle = \sum_{y \in Y} c_y \langle e_y, f_x \rangle.$$ 

Let $B = \sup_{y \in Y} \|\langle e_y, f_x \rangle\|$. Notice, $f_x$ is bounded for each $x$:

$$|f_x(f)| = |\sum_{y \in Y} c_y \langle e_y, f_x \rangle| = \|Ac\|_{\ell^p(X)} \leq \|c\|_{\ell^p(Y)} \leq \beta \|f\|_{H^p}.$$ 

If $H^p \not\subseteq H$, define $f_x$ as above for $f = \sum_{y \in Y} c_y e_y$, $\text{supp} \ c < \infty$. $f_x$ is still a bounded linear functional. Then by a corollary of the Hahn Banach theorem, we can extend $f_x$ to a bounded linear functional on the Banach space $H^p$ complete in the norm $\|\cdot\|_{H^p}$.

Now let $f \in H^p$, then there is a Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ that converges to $f$, where $f_n = \sum_{y \in Y} c_y e_y$, $\text{supp} \ c' < \infty$. This limit is unique. The boundedness of the linear functional gives us that $(f_x(f_n))_{n \in \mathbb{N}}$ is a Cauchy sequence which converges to $f_x(f)$ and this limit is also unique.

(a) Let $C$ be the coefficient operator, $Cf := \langle \langle f, f_x \rangle \rangle_{x \in X}$. First consider $f = \sum_{y \in Y} c_y e_y \in H^p \subseteq H$, where $c \in \ell^p(Y)$ and $\|f\|_{H^p} \approx \|c\|_{\ell^p(Y)}$. Then $Cf = \sum_{y \in Y} c_y \langle e_y, f_x \rangle$. We have

$$\|Cf\|_{\ell^p(X)} = \|Ac\|_{\ell^p(X)} \leq \alpha \|f\|^p_{H^p},$$

where $A$ is defined as in Lemma 4.4, with $(Ac)_{x \in X} = \sum_{y \in Y} \langle e_y, f_x \rangle c_y$, and $m = 1$. Hence the coefficient operator $C$ is bounded from $H^p$ to $\ell^p(X)$.

If $H^p \not\subseteq H$, then we can define $C$ as above on finite sums. As $C$ is continuous, $C$ can be extended to a bounded linear operator on the completion.

(b) Let $c = (c_x)_{x \in X}$ be a finite sequence. Let the reconstruction operator $R$ be the synthesis operator:

$$Rc = D^{\ast} c = \sum_{x \in X} c_x f_x.$$
By a proof nearly identical to that of Proposition 2.4 in [7], we have that $R$ is bounded on $l^p(X)$ for $1 \leq p \leq \infty$. If $f \in \mathcal{H}^p \subseteq \mathcal{H}$,

$$R((f,f_x)_{x \in X}) = \sum_{x \in X} \left( \sum_{y \in Y} c'_y e_y, f_x \right) \tilde{f}_x = \sum_{x \in X} \left( \sum_{y \in Y} c'_y (e_y, f_x) \right) \tilde{f}_x = \sum_{y \in Y} c'_y e_y = f.$$ 

If $f \in \mathcal{H}^p \not\subseteq \mathcal{H}$, let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}^p$ converging to $f$, where $f_n = \sum_{y \in Y} c'_y e_y$, $\text{supp} c' < \infty$. Then $R((f_x(f_n))_{x \in X}) = f_n$. Recall, $f_x$ is a bounded linear functional, so $f_x(f_n)$ converges to $f_x(f)$. As the limits are unique, we have that $R((f_x(f))_{x \in X}) = f$.

(c) We have that

$$||Cf||_{l^p(X)} = ||(f,f_x)_{x \in X}||_{l^p(X)} \leq \alpha||f||_{\mathcal{H}^p}$$

and

$$||RCf||_{\mathcal{H}^p} = ||f||_{\mathcal{H}^p} \leq D||Cf||_{l^p(X)}.$$ 

Hence

$$\frac{1}{\alpha} ||(f,f_x)_{x \in X}||_{l^p(X)} \leq ||f||_{\mathcal{H}^p} \leq D||((f,f_x)_x)_{x \in X}||_{l^p(X)}.$$ 

Hence $F$ is automatically a Banach frame for $\mathcal{H}^p$. \hfill \square

If we add a weight $m$ and consider the weighted $l^p$ space $l^p_m$, we run into the problem of having to define $m$ for $X,Y$, and $G$. We deal with this problem by embedding $X,Y,$ and $G$ into a larger space $S$ (this generalizes the case where $X,Y,$ and $G$ are subsets of $\mathbb{R}^d$) and adding the extra condition that the maps $a_X : X \to G, a_Y : Y \to G$ are such that $|x - a_X(x)|, |y - a_Y(y)| \leq \mu$ for some $\mu > 0$.

**Theorem 4.6.** Suppose $\mathcal{F} = (f_x)_{x \in X}$ is a frame and $\mathcal{E} = (e_y)_{y \in Y}$ be a Riesz basis for $\mathcal{H}, \mathcal{F}, \mathcal{E} \in S^c, X,Y \subset S$, where $S$ is an abelian group. Assume $G \subset S$, and the maps $a_X : X \to G, a_Y : Y \to G$ are such that $|x - a_X(x)|, |y - a_Y(y)| \leq \mu$ for some $\mu > 0$. Let $m : S \to \mathbb{R}_+^*$ be a submultiplicative weight function, i.e., a non-negative, locally integrable function on $S$ such that for all $x,y \in S$, $m(x + y) \leq m(x)m(y)$. Assume without loss of generality that $m$ is continuous and symmetric. If $\mathcal{F} \sim \mathcal{E}$, then $\mathcal{F}$ is a Banach frame for the family of Banach spaces $\mathcal{H}^p_m$ associated to $\mathcal{E}$.

**Proof.** The proof is almost identical to that of the previous theorem. Notice, Lemma 4.5 still holds true if we replace $\mathbb{R}^d$ by $S$. \hfill \square

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