A New Perspective on Finding the Viewpoint

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Abstract: We take a fresh perspective on an old idea and create an alternate way to answer the question: where should we stand in front of an image in two-point perspective to view it correctly? We review known geometric and algebraic techniques, then use the cross ratio to derive a simple algebraic formula and a technique that makes use of slopes on a perspective grid.

Figure 1: Hendrick van Vliet, Interior of the Oude Kerk, Delft, 1660

You are in the Metropolitan Museum of Art in New York, and you come across Hendrick van Vliet's *Interior of the Oude Kerk, Delft*, 1660 [14]. Through his skilled use of perspective, van Vliet seems eager to make you feel as though you were actually there in the Oude Kerk, the oldest building still standing in Amsterdam. A perspective painting done with mathematical accuracy will act like a window to the 3-dimensional world, but you need to be standing where the artist stood to get this window effect.

As Leonardo da Vinci wrote in his notebooks, the spectator will see "every false relation and disagreement of proportion that can be imagined in a wretched work, unless the spectator, when he looks at it, has his eye at the very distance and height and direction where the eye or the point of sight was placed in doing this perspective" (543, [9]). So how do we find this "point of sight"? Your instinct may be to stand directly in front of the center of the van Vliet painting. However, according to our measurements, you will need to stand with your eye about 2/3 of the way down far over to the left, nearly at the edge of the painting, back about the length of the height of the painting.

Although the painting is still realistic and impressive when viewed from other locations, without knowledge of the "point of sight" more commonly known as the *viewpoint*, it would be difficult to experience the spacious depth of the centuries-old church. Thus, we offer this article as a proposal to curators and museums to consider identifying the viewpoints of several perspective paintings in their collections and present the information in a format that helps visitors appreciate the art from the intended angle. We review several known geometric methods, simplify a known algebraic method and use the insights gained from the simplification to introduce a new method which we call the perspective slope method, a satisfying blend of geometric and algebraic techniques.

1 Background

To determine the odd viewpoint for van Vliet's painting, we first notice that two main sets of lines parallel in the Oude Kerk appear to go to two different vanishing points, to the left and right of the painting as in Figure 2. This tells us that the painting is done in *two-point perspective*. Compared to a painting done in one-point perspective, this makes our job considerably harder (for more on finding the viewpoint of one-point perspective drawings along with the mathematics of perspective drawing, we recommend Frantz and Crannell's Viewpoints [5]).

Over the centuries, several mathematicians have solved the problem of finding the viewpoint for two-point perspective for special cases, Simon Stevin (1605), Johann Heinrich Lambert (1759) and most notably, Brook Taylor of Taylor series fame, who gave several solutions in his two books on linear perspective (1715, 1719). A readable account of the history of what is known as the inverse problem of perspective is found in Andersen's book on the history of mathematical perspective [1]. A more modern, related problem in computational projective geometry involves determining the exact location of a camera using on-site measurements and clues given in a photograph, called "camera calibration" or "camera resectioning". Mathematics Magazine devotees may remember two articles written on this topic, "Where the Camera Was" by Byers

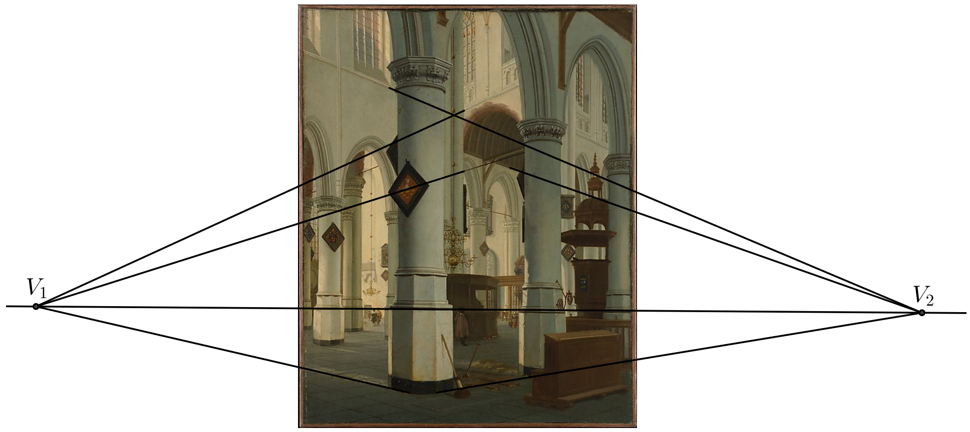

Figure 2: Two-point perspective

and Henle [2] and "Where the Camera Was, Take Two" by Crannell [3], which respectively discussed an algebraic and a geometric approach to the problem.

2 The geometric methods

The standard geometric method described in [5] uses semi-circles to find the viewpoint, as shown in Figure 3. There is no need for measurement or computation, but as we shall see, this method can sometimes be impractical.

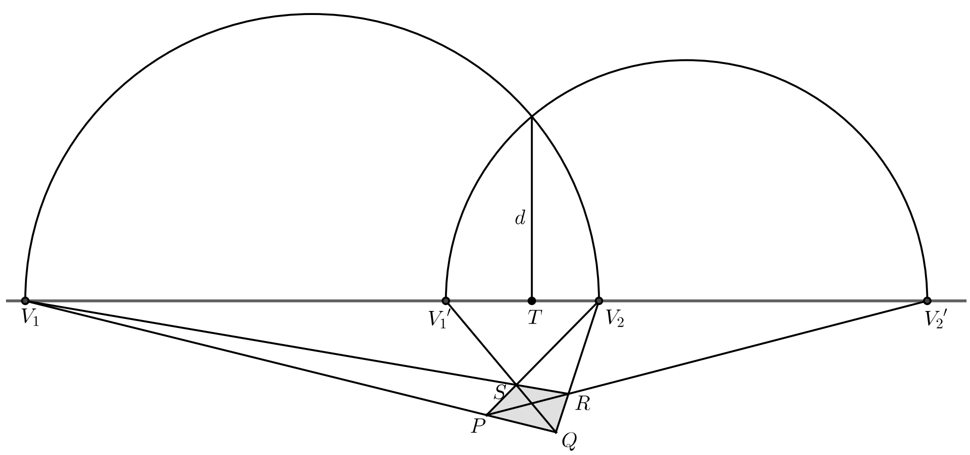

Figure 3: Finding the viewpoint for two point perspective

The first step is to identify a four-sided figure in the painting, such as $PQRS$

in Figure 3, which we know or can reasonably assume to be a square on the ground or in a plane parallel to the ground drawn in perspective. The two pairs of opposite sides of the square are parallel in the real world and not parallel to the canvas "window", which we call the *picture plane*, so their perspective images intersect at the *principal vanishing points* V_1 and V_2 along the *horizon line*. The diagonals of our square are also not parallel to the picture plane, and so their perspective images have vanishing points V_1' and V_2' along the horizon line.

The next step is to draw two semi-circles, one with diameter $v = \overline{V_1V_2}$ and another with diameter $v' = \overline{V_1'V_2'}$, and find their intersection point. Through that intersection point, we draw a line orthogonal to the horizon line. The viewpoint is then determined by placing your eye directly in front of T as shown in Figure 3 at a distance d from the painting.

To understand why this works, let's float above the scene, to get the bird's eye view. We would see the viewer (or the viewer's eye) at O, the picture plane seen as a horizontal line, and the undistorted square, as in Figure 4. We can now see the distance from the viewer to the picture plane d and the point T on the picture plane directly in front of the viewer's eye.

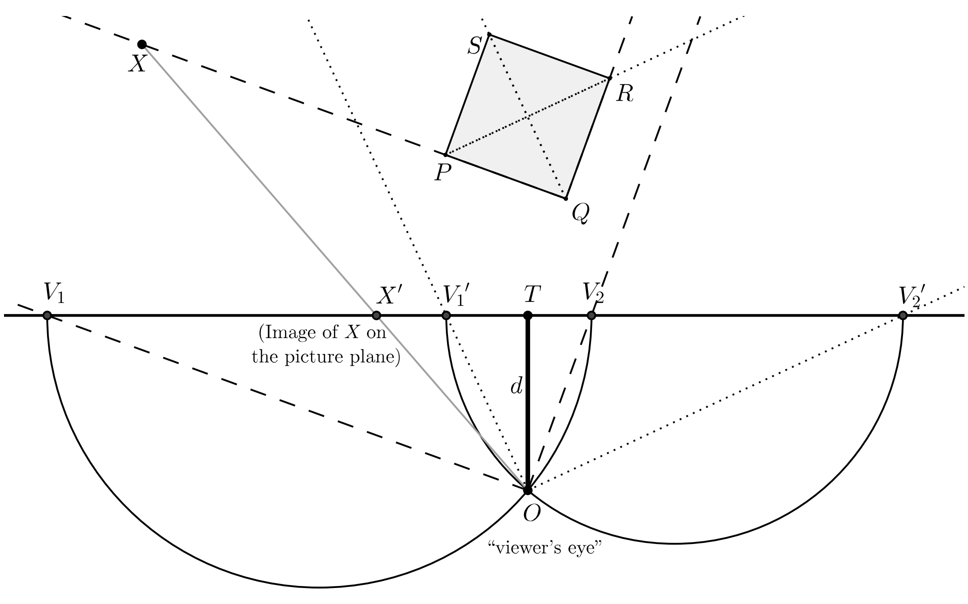

Figure 4: Bird's eye view

V_1 is the vanishing point of the line PQ. To understand how to find V_1 on this top view, consider a point X on line PQ. Its image X' is located on the picture plane where the sight line from the viewer's eye O to the point X intersects the picture plane. As we pull X off towards the left, X' is also pulled along the line of the picture plane towards the left. Notice, X' will converge to the point where the viewer can no longer see the line PQ, i.e., where the line parallel to PQ intersects the picture plane. So this must the the vanishing point V_1. Similarly, we can find the other vanishing points.

Since the sight lines to the principal vanishing points (indicated as the dashed lines in Figure 4) form a right angle as do the sight lines to the diagonal vanishing points (indicated as the dotted lines in Figure 4), we can use Thales' theorem for triangles inscribed in a semi-circle to draw two semi-circles between the vanishing points. We find the viewer's eye at the intersection of the semi-circles, which determines the viewing distance d and viewing target T.

So in summary, the standard geometric method is as follows:

1. Find the vanishing points V_1, V_2, V_1' and V_2' along the horizon line.

2. Draw the semi-circles with diameters $v = \overline{V_1V_2}$ and $v' = \overline{V_1'V_2'}$.

3. Find the intersection of the semi-circles, and drop a line down perpendicular to the horizon line to find T, the point that should be directly in front of your eye.

4. The distance between the intersection and the horizon line is d, how far back from the painting you should stand.

Generally, this is a good method. Finding a square in the painting can sometimes be easy, for example, if there is a tiled floor. In van Vliet's painting, it's more difficult. The floor is tiled, but we know that the tiles in the Oude Kerk are not square (it should be remarked that all of these techniques generalize to a more general parallelogram situation, but you will need to know the angles and ratios of lengths). We decided to use the base of the front column, which we can reasonably assume is a square with the corners cut off for the following reason. The walls of the Oude Kerk are at 90 degree angles, and the two lines along the base vanish at the same vanishing points as the perpendicular lines along the walls as seen in Figure 2, hence are perpendicular themselves. We can reasonably assume that the columns have a circular rather than an oval cross section, so we can conclude that the base is square with the corners cut off. If we apply the geometric method to the van Vliet painting using this square, we find that V_2' is located to the left of the painting, and it is quite a distance away from the other vanishing points as you can see in Figure 5.

Taylor [10, 1] and Lambert [8, 1] provide two alternatives, which do not require the distant V_2' vanishing point. As shown in Figure 6, Taylor suggests we draw two right isosceles triangles with hypotenuses $\overline{V_1V_1'}$ and $\overline{V_1'V_2}$ respectively then draw circles using the apexes as the centers and the legs of the triangles as the radii. The intersection of these two circles will be exactly where the intersection of the semi-circles was in the previous method, thus giving us T and d. If you are able to draw right isosceles triangles and circles, this is not a bad alternative. Lambert gives us yet another alternate method, as shown in Figure 7. Again, it only requires three vanishing points, however, you have to be able to determine where the lines form a 45 degree angle.

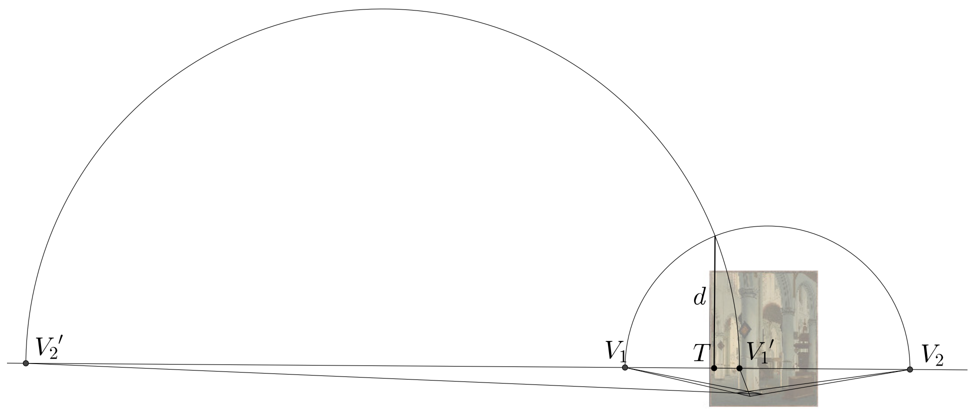

Figure 5: The standard geometric method for van Vliet's painting

3 The algebraic method

Since we can find T and d by finding the intersection of two circles, we can surely find an algebraic formula for T and d. Indeed, this was done by Greene [6] who came up with a rather complicated formula which we will describe at the end of this section. We provide here a simplified version of Greene's formula whose setup and derivation provide insight into our new perspective slope method. For details on the calculations, we refer readers to the supplementary document on Mathematics Magazine's website.

In Figure 8, we superimpose a coordinate system with the horizon line as the x-axis in order to describe the semi-circles algebraically. The ratio of the distances between the principal vanishing points and the diagonal vanishing point between them becomes quite important, so we shall denote this ratio as $\rho = \frac{v_L}{v_R}$, where v_L is the distance on the left (between V_1 and V_1') and v_R is the distance on the right (between V_1' and V_2).

Theorem 1. *Let t be the distance from the left-most principal vanishing point to the viewing target and d the viewing distance. Then*

$$t = \frac{\rho^2 v}{\rho^2 + 1} \qquad \textit{and} \qquad d = \frac{\rho v}{\rho^2 + 1} = \frac{t}{\rho},$$

where $\rho = \frac{v_L}{v_R}$ and $v = v_L + v_R$.

Proof. Consider Figure 8. The equations for the semi-circles are as follows.

$$\left(x - \frac{v}{2}\right)^2 + y^2 = \left(\frac{v}{2}\right)^2 \qquad \text{and} \qquad \left(x - \left(v_L + \frac{v'}{2}\right)\right)^2 + y^2 = \left(\frac{v'}{2}\right)^2$$

To find t, the distance to the viewing target, we need to find the x-value of the circles' intersection point, so we subtract one equation from the other and

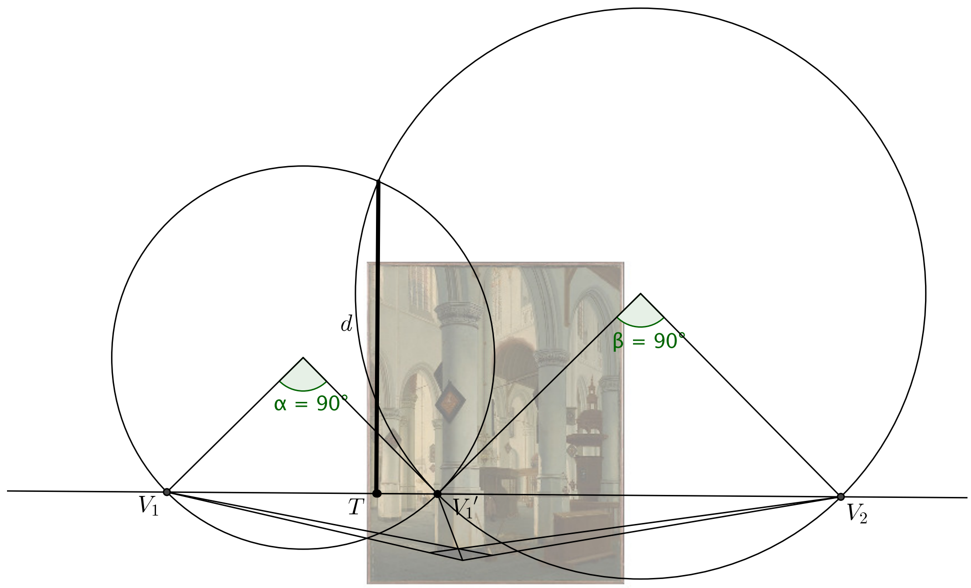

Figure 6: Taylor's method

solve for x. We find that

$$x = \frac{v_L(v_L + v')}{v_L - v_R + v'}. \tag{1}$$

In order to write (1) entirely in terms of v_L and v_R, we use what is known as the cross ratio. A *cross ratio* of four points along a line, A, B, C and D is the following product of ratios of directed distances,

$$\times(ABCD) = \frac{|AB|}{|BC|}\frac{|CD|}{|DA|},$$

where a distance becomes directed by choosing a direction for a line, for example, a positive direction from A to B, so that the directed distance $|AB|$ is positive and $|BA|$ is negative. There are some very nice properties of the cross ratio, most notably that it is invariant under projections. For more on cross ratios, see [4]. The property that is most important for us here is that $\times(V_1V_1'V_2V_2') = -1$.

To see this, notice that the four-sided figure $PQRS$ along with its two pairs of opposite sides and its diagonals (the third pair of opposite sides) form what is known as a *complete quadrangle*, with three *diagonal points* V_1, V_2 and the intersection of the diagonals. By definition, a set of four collinear points is a *harmonic set* if there exists a complete quadrangle such that two of the points are diagonal points and the other two points are on the opposite sides determined by the third diagonal point. Hence the two principal vanishing points and the two diagonal vanishing points form a harmonic set, denoted $H(V_1, V_2; V_1', V_2')$. It is well known that the cross ratio of a harmonic set equals -1. Thus, we have

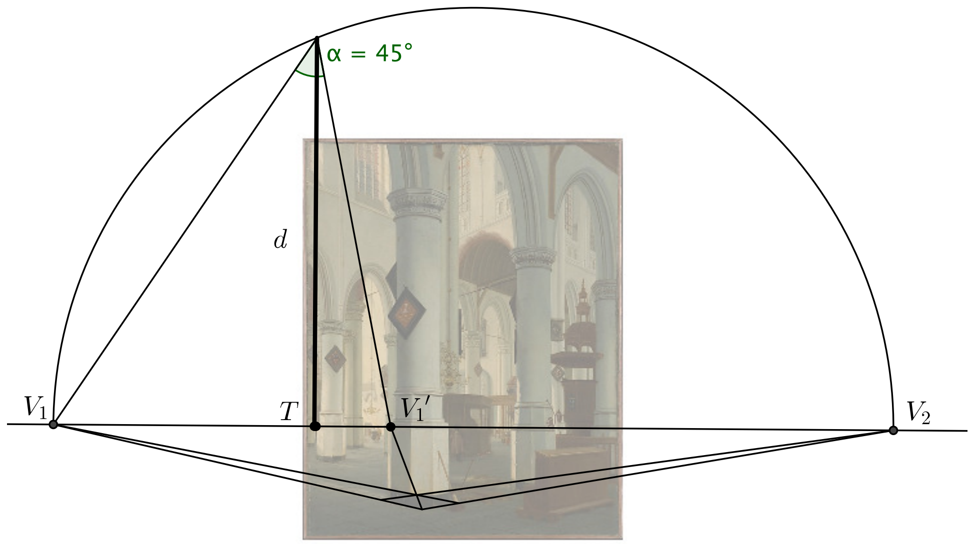

Figure 7: Lambert's method

$$\times(V_1V_1'V_2V_2') = \frac{|V_1V_1'|}{|V_1'V_2|}\frac{|V_2V_2'|}{|V_2'V_1|} = \frac{v_L}{v_R}\cdot\frac{v'-v_R}{-v'-v_L} = -1. \tag{2}$$

Solving for v' in (2), we find

$$v' = \frac{2v_Lv_R}{v_L - v_R}. \tag{3}$$

Going back to our equation (1), substituting in (3) gives us

$$x = \frac{v_L\left(v_L + \frac{2v_Lv_R}{v_L-v_R}\right)}{v_L - v_R + \frac{2v_Lv_R}{v_L-v_R}} = \frac{v_L^2(v_L+v_R)}{v_L^2+v_R^2} = \frac{(\frac{v_L}{v_R})^2(v_L+v_R)}{(\frac{v_L}{v_R})^2+1} = \frac{\rho^2 v}{\rho^2+1} = t.$$

Solving for the y-value of the intersection point gives us d.

$$y^2 = \left(\frac{v}{2}\right)^2 - \left(\frac{\rho^2 v}{\rho^2+1} - \frac{v}{2}\right)^2 = \frac{v^2}{4}\left(\frac{2\rho^2}{\rho^2+1}\right)\left(\frac{2}{\rho^2+1}\right) = \frac{v^2\rho^2}{(\rho^2+1)^2}.$$

Taking the positive solution, we get

$$d = \frac{v\rho}{\rho^2+1} = \frac{t}{\rho}.$$

□

The algebraic method is as follows:

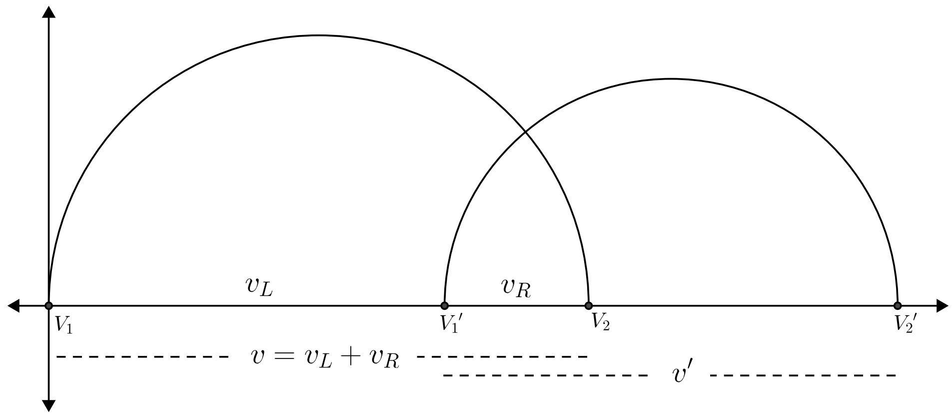

Figure 8: The algebraic method

1. Find the vanishing points V_1, V_2 and V_1' along the horizon line.
2. Measure v_L and v_R, and calculate $\rho = \frac{v_L}{v_R}$ and $v = v_L + v_R$.
3. Calculate $t = \frac{v\rho^2}{\rho^2+1} = |V_1 T|$ to find the viewing target T.
4. Divide t by ρ to find the viewing distance d.

Applying this to the van Vliet painting, $\rho = \frac{69.44}{103.44} \approx 0.67$, $v = 69.44 + 103.44 = 172.88$. Thus $t \approx 53.71$ cm and $d \approx 80.00$ cm. In Figure 9, we see where this T is located on the picture plane, and we see that it nearly coincides with the T found using the geometric method. The small difference of approximately 0.34 cm is due to round-off error. We can make the calculation easier, by approximating $\rho \approx 2/3$ and $v \approx 173$, and the simple calculation $173 \cdot 4/13 \approx 53.23$ is still less than a centimeter off.

We compare this with Greene's formula given in [6],

$$t = \frac{vv_L^2}{v^2 - 2vv_L + 2v_L^2}\ ,\ d = \frac{[vv_L^2(v^3 - 2v^2 v_L + vv_L^2)]^{1/2}}{v^2 - 2vv_L + 2v_L^2}.$$

Instead of ρ and v, he used v_L and v (denoted s and D in [6]). This version of the formula is considerably harder to remember and harder to use.

4 The perspective slope method

We come now to our new method, which we call the perspective slope method. Earlier, we mentioned that the cross ratio of a harmonic set equals -1. Let's see what happens when we replace the diagonal vanishing point V_1' with T.

$$\times(V_1 T V_2 V_2') = \frac{|V_1 T|}{|TV_2|}\frac{|V_2 V_2'|}{|V_2' V_1|} = \frac{t}{t - v_L} \cdot \frac{v' - v_R}{v' + v_L}. \qquad (4)$$

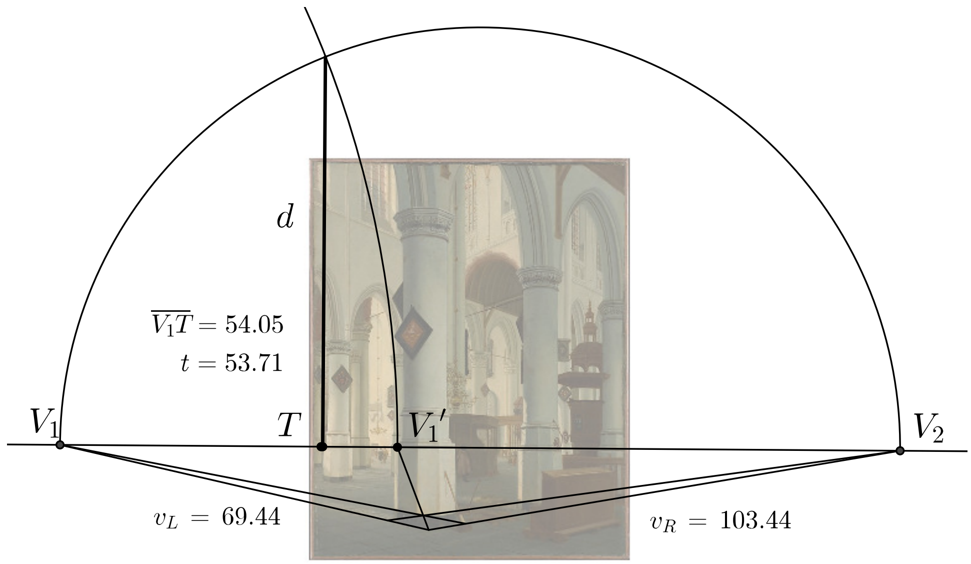

Figure 9: The algebraic method applied to Van Vliet's painting

By (2), $\frac{v'-v_R}{v'+v_L} = -\frac{1}{\rho}$. So using this substitution along with the formula for t in (4), we find

$$\times(V_1TV_2V_2') = \frac{\frac{\rho^2 v}{\rho^2+1}}{v - \frac{\rho^2 v}{\rho^2+1}} \frac{-1}{\rho} = \frac{\rho^2 v}{\rho^2 v + v - \rho^2 v} \frac{-1}{\rho} = -\rho.$$

This is rather nice, so how might we use it?

To answer this question, we establish a relationship between cross ratios and the slopes of lines on a coordinate grid drawn in perspective. Assuming $PQRS$ is the perspective image of a square, we can use this to create a perspective image of a coordinate grid with $PQRS$ as the image of one square of the grid. Assuming V_1 is the vanishing point of the x-axis and V_2 the vanishing point of the y-axis, we give a sketch of the proof below that a line with slope m in the perspective coordinate grid will vanish at a point M such that $\times(V_1MV_2V_2') = m$.

Theorem 2. *Let $PQRS$ be a complete quadrangle with associated harmonic set $H(V_1, V_2; V_1', V_2')$. Assuming that $PQRS$ is the perspective image of a square with the perspective coordinate grid set up as described above, a line with slope m in the perspective coordinate grid will vanish at a point M such that $\times(V_1MV_2V_2') = m$.*

Sketch of Proof.

We first look at the one-point perspective situation, as in Figure 10. ℓ will vanish at M located d/m away from V_2. This is clear from considering the top view, but details are found in [5]. Hence, the cross ratio will be

$$\times(V_1 M V_2 V_2') = \frac{|V_1 M|}{|M V_2|}\frac{|V_2 V_2'|}{|V_2' V_1|} = \frac{\infty}{-d/m}\frac{d}{-\infty} = m.$$

Since the cross ratio is a projective invariant, this will also hold in two point perspective.

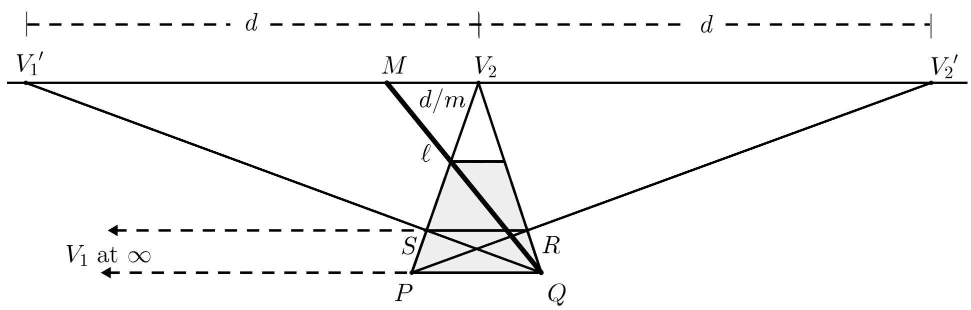

Figure 10: One point perspective situation

So in order to find T, we need only calculate $\rho = \frac{v_L}{v_R}$, draw the line with slope $-\rho$ and its vanishing point will be T! Once you determine the location of T and measure $t = |V_1 T|$, $d = t/\rho$.

So our final method is as follows:

1. Find the vanishing points V_1, V_2 and V_1' along the horizon line.
2. Measure v_L and v_R, and calculate $\rho = \frac{v_L}{v_R}$.
3. Use your perspective drawing skills to draw a line with slope $-\rho$ in perspective, and find its vanishing point. This is T.
4. Divide $t = |V_1 T|$ by ρ to find the viewing distance d.

In van Vliet's painting, we approximate $\rho = 0.67$ by $2/3$, which makes finding the slope a bit easier. One way is to draw a 2×3 grid, as shown in Figure 11. Despite our rough estimation, it does a pretty good job! The viewing distance is then roughly $3/2$ of t. In a two-point perspective painting with a tiled floor, this method is very easy to use.

5 Conclusion

We have presented a variety of methods for finding the viewpoint of a two-point perspective painting, each with their advantages and disadvantages. We have also found that in many of our calculations, we used approximations that shifted our viewpoint by a small amount. Do we have to stand with our eye "at the

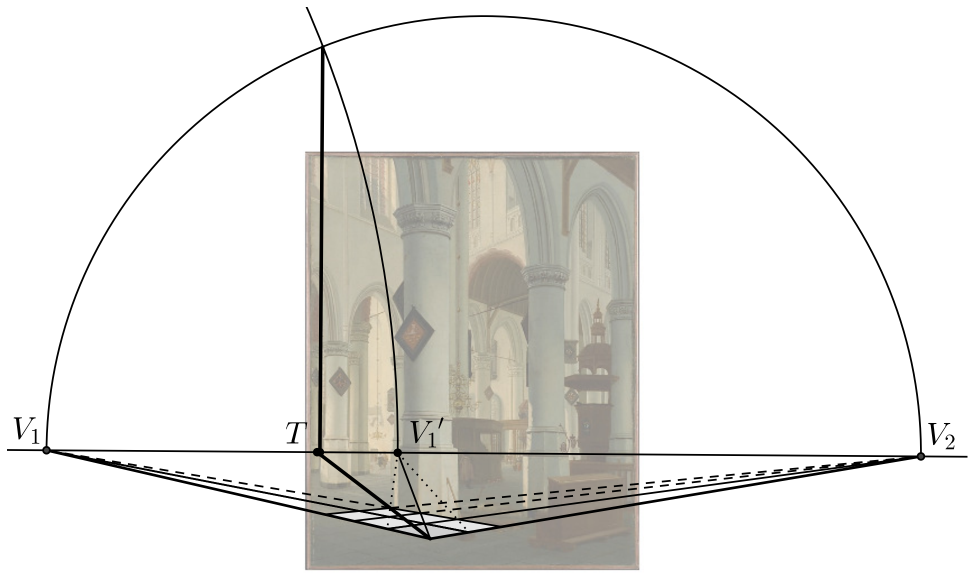

Figure 11: The perspective slope method applied to the van Vliet drawing

very distance and height and direction" as Leonardo da Vinci suggests or do we have a little room for error? Since the Renaissance, much research has been done on the psychology of human vision and picture perception and the answer is more nuanced than previously thought. Regarding being too close or too far from the picture plane, a simple geometrical analysis of the situation tells us that the distortion is proportional to the viewing distance so we have more room for error if the viewing distance is large. It was found in one experiment of 12 college students in [12] that on average, they perceived a 1:1 ratio of sides of a rectangle in perspective at 21.4 cm, although the actual viewing distance was 28 cm. This was found to not be significantly different from the mean with a standard deviation of 16.7 cm. And although the math seems to indicate that we would perceive a 2:1 ratio of sides at double the viewing distance, the same study showed that actually this was perceived at an average of 422.1 cm with standard deviation 92.3 cm. In another study [7], it was shown that we compensate for a difference in viewing distance better if the painting does not have a low eye height and does not depict a wide-angle view. In fact, even at the correct viewpoint, we will see distortions around the periphery in a mathematically accurate, wide-angle perspective drawing. And finally, if we are to the left or right of the viewpoint or if we view the painting at an angle, having both eyes open allows us to compensate for a larger difference [13].

If our eye is at the viewpoint, the research generally supports the idea that we will have the feeling of being immersed in a 3D environment. Indeed, if you try this with the van Vliet painting, you see the interior from the height of an average person standing in the church, and you become suddenly aware of the people and chandelier in the far room. The arches soar overhead and you can feel the spaciousness of the old church. The effect is magical. Our great hope is that you will share this article with your local museums or at the very least feel empowered to use these techniques yourself on digital images to determine viewpoints prior to a museum visit.

6 Acknowledgment

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7 Biosketch

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