Desargues’ Theorem
Working toward a proof

In what follows, we will use four axioms of projective geometry that happen to hold true for \( \mathbb{R}P^3 \). (You proved for homework that the first axiom holds; proving the second through fourth statements take more linear algebra than I wanted to go into in this class).

**Two Point Axiom.** Any pair of distinct points in \( \mathbb{R}P^3 \) lie on a common line.

**Two Line Axiom.** A pair of distinct lines in \( \mathbb{R}P^3 \) intersect in a single point if and only if they lie in a common plane.

**Two Plane Axiom.** A pair of distinct planes in \( \mathbb{R}P^3 \) intersect in a single line.

**Line-Plane Axiom.** A plane and a line not contained in that plane in \( \mathbb{R}P^3 \) intersect in a single point.

We will also use the following helpful definition of a triangle.

**Definition.** In projective geometry, we say a triangle is a mesh containing exactly three points and three lines\(^1\).

For each [T/F] item below, decide whether the statement is true or false. Use the definition and axioms above to explain your answer.

1. Suppose two triangles \( \mathcal{T} \) and \( \mathcal{T}' \) are perspective from a line \( \alpha \). That is, suppose the lines \( r, y, b \in \mathcal{T} \) and \( r', y', b' \in \mathcal{T}' \) meet pairwise at the three points \( R, Y, B \). (Think “red, yellow, blue”). What can you say about the three points \( R, Y, \) and \( B \)? Where do they lie?

\(^1\)For homework (see the next page), you will look at the question of whether we need this much structure (that is three points, AND three lines, AND that they form a mesh) in order to prove Desargues’ Theorem. But in what follows, we’ll assume we’re working with a mesh structure.
We’ll continue to color-code our points and lines using orange, purple, and green. Suppose $N = r \cdot y$, $P = r \cdot b$, and $G = y \cdot b$. (Same for the points in $T'$.)

2. [T/F] It must be true that $y$ and $y'$ lie in a common plane (say, $\pi_{yellow}$).

3. If the above statement is true, make similar claims about $\pi_{red}$ and $\pi_{blue}$. If the statement above is false, fix it.

4. [T/F] $GG'$ might be the empty set.

5. [T/F] $GG'$ is a line.

6. [T/F] $GG'$ is a point.

7. In which of the above plane(s) — if they exist — does $GG'$ lie? Explain your answer using the axioms above.

8. [T/F] $GG'$ and $NN'$ must lie in a common plane. (If so, which plane(s)? If not, why not?)

9. [T/F] $GG'$ is necessarily incident with $NN'$.

10. The point $GG' \cdot NN'$ — if it exists — lies in which plane(s)? Must it lie in $\pi_{yellow}$? $\pi_{red}$? $\pi_{blue}$?

11. [T/F] If the triangles $T$ and $T'$ lie in different planes, then the line $PP'$ lies in the plane $\pi_{yellow}$.

12. [T/F] If the triangles $T$ and $T'$ lie in different planes, then the three planes $\pi_{yellow}$, $\pi_{red}$, and $\pi_{blue}$ meet in a single point.

13. [T/F] If the triangles $T$ and $T'$ lie in different planes and are perspective from a line, then they are perspective from a point.

14. Why does this set of questions and answers fail to prove statement 13 for two triangles in the same plane? Which of the above statements/questions would have a different answer in that situation?

Homework proofs: For each of the statements below, either prove the statement or provide a counter-example to show that it is false.

P6. Given seven distinct points $O$, $A$, $B$, $C$, $A'$, $B'$, and $C'$ so that $O$ is coincident with $AA'$, $BB'$, and $CC'$, then it must follow that the three points $AB \cdot A'B'$, $BC \cdot B'C'$, and $CA \cdot C'A'$ are colinear.

P7. Given seven distinct lines $\alpha$, $j$, $k$, $l$, $j'$, $k'$, and $l'$ so that $\alpha$ is coincident with $j \cdot j'$, $k \cdot k'$, and $l \cdot l'$, then it must follow that the three lines $(j \cdot k)(j' \cdot k')$, $(k \cdot l)(k' \cdot l')$, and $(l \cdot j)(l' \cdot j')$ are coincident—that is, they meet in a single point.
The remainder of the proof of Desargues’ Theorem follows from these two propositions, which you should prove in homework groups and hand in.

P8. Given two meshes $M$ and $N$, both visible from the origin. Suppose that these meshes are perspective from a point. Then their images $M'$ and $N'$ are likewise perspective from a point.

P9. If the triangles $T$ and $T'$ are perspective from a point, then they are perspective from a line.

Use propositions 8 and/or 9, together with the statement 13, to complete the following questions. We begin now with two triangles $T$ and $S$ lying in the same plane (we can assume this plane is $z = 1$). Suppose $T$ and $S$ are perspective from a line $\alpha$. Recall that we use the symbol $'$ to denote the projection of an object onto the plane \{ $z = 1$ \}.

14. For our proof, we’ll want to draw a triangle $U$ in another plane $\pi$, so that $T$ and $U$ are perspective from a line. How might we want to choose this new plane $\pi$? (There are lots of possible answers to this question, of course. If you don’t like your answers to subsequent questions, come back and fix this so that the next few answers are easier).

15. Having chosen this plane, let $U$ be the triangle in $\pi$ that is perspective from the point $(0, 0, 0)$. This means that $U' = [$ \phantom{1} $]$?

16. What is the name of the line that $U$ and $T$ are perspective from?

17. If we chose $\pi$ correctly in step 14, then $U$ should also be perspective with $S$ from a line. How do we know this is so?

18. Which statement above tells us that, therefore, $S$ and $U$ are perspective from a point?

19. Where is/what is $S'$?

20. Put the pieces together. We know $S$ and $U$ are perspective from a point. Why does this imply the very thing we want to prove: that $S$ and $T$ are perspective from a point?