Finding the Viewpoint at a Museum: A How-To Guide

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Figure 1 Hendrick van Vliet, Interior of the Oude Kerk, Delft, 1660

You are in the Metropolitan Museum of Art in New York, and you come across Hendrick van Vliet's, Interior of the Oude Kerk, Delft, 1660 [11].

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Through his skilled use of perspective, van Vliet seems eager to make you feel as though you were actually there in the Oude Kerk, the oldest building still standing in Amsterdam. A perspective painting done with mathematical accuracy will act like a window to the 3-dimensional world, but you need to be standing exactly where the artist stood to get this window effect.

As Leonardo da Vinci wrote in his notebooks, the spectator will see "every false relation and disagreement of proportion that can be imagined in a wretched work, unless the spectator, when he looks at it, has his eye at the very distance and height and direction where the eye or the point of sight was placed in doing this perspective" (543, [8]). How do we find this "point of sight"? Your instinct may be to stand directly in front of the center of the painting. However, according to our measurements, you will need to stand with your eye about 2/3 of the way down far over to the left, nearly at the edge of the painting, back about the length of the height of the painting.

Our goal in this article is to provide techniques that allow museum-goers to find the "point of sight", more commonly known as the *viewpoint*, with only a few tools: pencil, paper and ruler. We will also assume that we are able to find a quadrangle we know to be a square on the ground in the painting. We quickly review a few known geometric methods, then give a simple algebraic method. We then present a new method, which we call the perspective slope method, a satisfying blend of geometric and algebraic that helps us practice our perspective drawing skills.

Background

To determine the odd viewpoint for van Vliet's painting, we first notice that two main sets of lines parallel in the Oude Kerk appear to go to two different vanishing points, to the left and right of the painting as in Figure 2. This tells us that the painting is done in *two point perspective*. Compared to a painting done in one point perspective, this makes our job considerably harder (for more on the mathematics of perspective drawing, we recommend Frantz and Crannell's Viewpoints [5]).

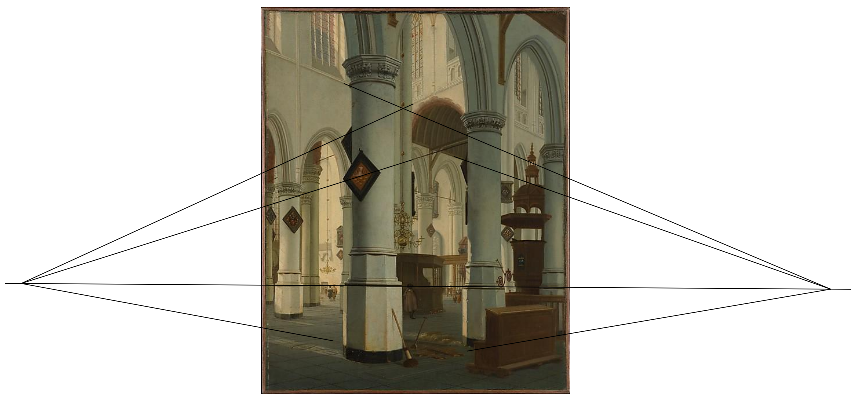


Figure 2 Two point perspective

Over the centuries, several mathematicians have solved the problem of finding the viewpoint for two point perspective for special cases, Simon Stevin (1605), Johann Heinrich Lambert (1759) and most notably, Brook Taylor of Taylor series fame, who gave several solutions in his two books on linear perspective (1715, 1719). A readable account of the history of what is known as the inverse problem of perspective is found in Andersen's book on the history of mathematical perspective [1]. A more modern version of the problem in computational projective geometry involves determining the exact location of a camera by clues given in a photograph, called "camera calibration" or "camera resectioning". Mathematics Magazine devotees may remember two articles written on this topic, "Where the Camera Was" by Byers and Henle [2] and "Where the Camera Was, Take Two" by Crannell [3], which respectively discussed an algebraic and a geometric approach to the problem. Unfortunately, these more modern methods are inappropriate for the museum-goer. The computational projective geometers use matrices, which requires too much computation to be done by paper and pencil, and Byer and Henle and Crannell's methods require on-site measurements, which we are unable to do. So we will start by revisiting the older geometric methods.

The Geometric Methods

The standard geometric method described in [5] uses semi-circles to find the viewpoint, as shown in Figure 3. There is no need for measurement or computation, but as we shall see, this method can sometimes be impractical.

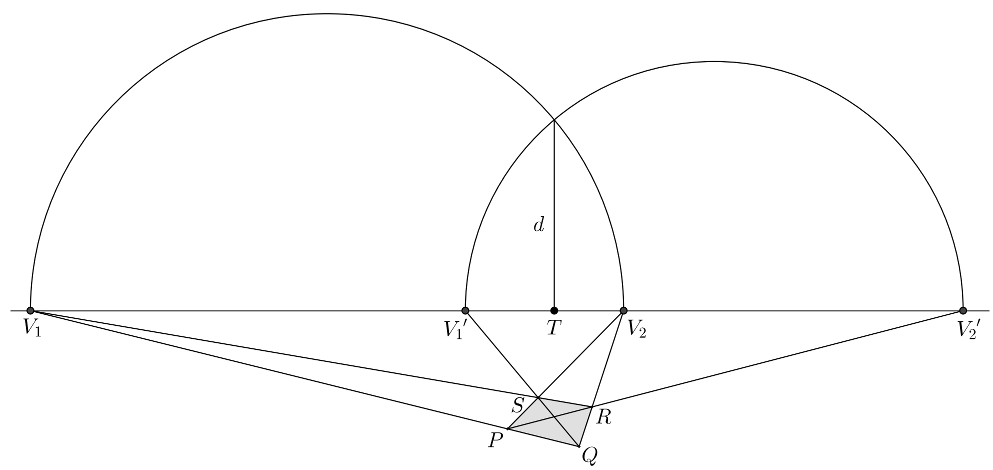


Figure 3 Finding the viewpoint for two point perspective

The first step is to identify a quadrangle in the painting, such as $PQRS$ in Figure 3, and assume it is a square drawn in perspective. The two pairs of opposite sides of the square are parallel in the real world and not parallel to the picture plane so they intersect at the *principal vanishing points* V_1 and V_2 along the *horizon line*. The diagonals of our square are also not parallel with the picture plane, and so we can find their vanishing points V_1' and V_2' along the horizon line.

The next step is to draw two semi-circles, one with diameter $v = \overline{V_1V_2}$ and another with diameter $v' = \overline{V_1'V_2'}$, and find their intersection point. Through that intersection point, we draw a line orthogonal to the horizon line. The viewpoint is then determined by placing your eye directly in front of T as shown in Figure 3 at a distance d from the painting.

To understand why this works, let's float above the scene, to get the bird's eye view. We would see the viewer (or the viewer's eye) at O, the picture plane seen as a horizontal line and the undistorted square, as in Figure 4. We can now see the distance from the viewer to the picture plane d and the point T on the picture plane directly in front of the viewer's eye.

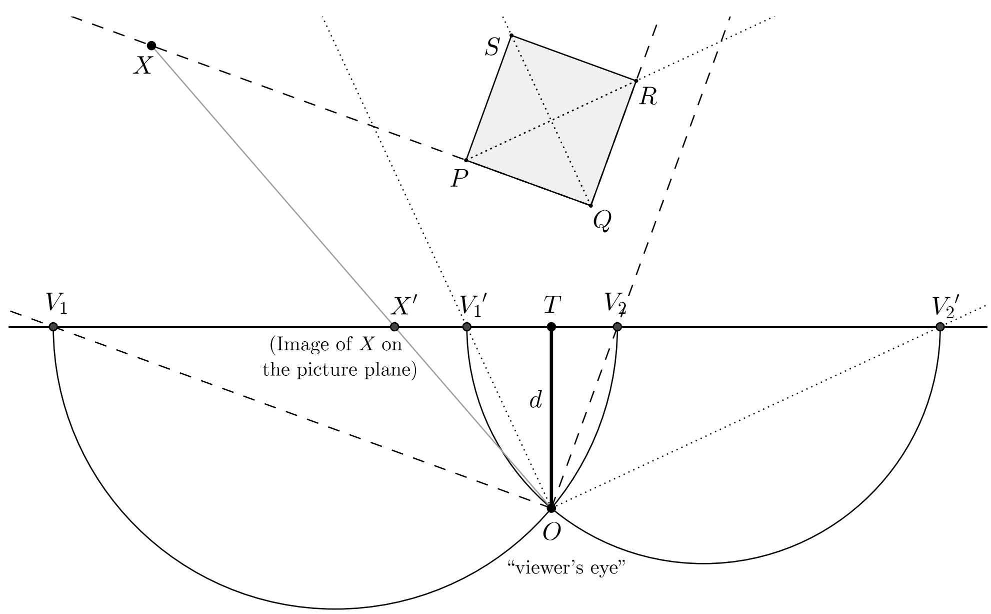


Figure 4 Bird's eye view

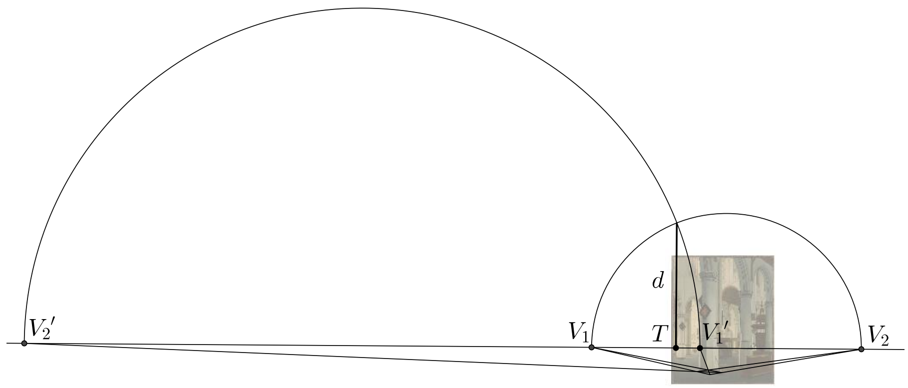
V_1 is the vanishing point of the line PQ. To understand how to find V_1 on this top view, consider a point X on line PQ. Its image X' is located on the picture plane where the sight line from the viewer's eye O to the point X intersects the picture plane. As we pull X off towards the left, X' is also pulled along the line of the picture plane towards the left. Notice, X' will converge to the point where the viewer can no longer see the line PQ, i.e., where the line parallel to PQ intersects the picture plane. So this must the the vanishing point V_1. Similarly, we can find the other vanishing points.

Since the sight lines to the principal vanishing points (indicated as the dashed lines in Figure 4) form a right angle as do the sight lines to the diagonal vanishing points (indicated as the dotted lines in Figure 4), we can use Thales' theorem to draw two semi-circles between the vanishing points. We find the viewer's eye at the intersection of the semi-circles, which determines the viewing distance d and viewing target T.

So in summary, the standard geometric method is as follows:

1. Find the vanishing points V_1, V_2, V_1' and V_2' along the horizon line.
2. Draw the semi-circles with diameters $v = \overline{V_1V_2}$ and $v' = \overline{V_1'V_2'}$.
3. Find the intersection of the semi-circles, and drop a line down perpendicular to the horizon line to find T, the point that should be directly in front of your eye.
4. The distance between the intersection and the horizon line is d, how far back from the painting you should stand.

Generally, this is a good method. Finding a square in the painting can sometimes be easy, for example, if there is a tiled floor. In van Vliet's painting, it's more difficult. The floor is tiled, but they are not square (it should be remarked that all of these techniques generalize to a more general parallelogram situation, but you will need to know the angles and ratios of lengths). We used the octagonal column, which we can assume has a square base with the corners cut off. Although the diagonal vanishing point V_1' is always between the principal vanishing points V_1 and V_2, the other diagonal vanishing point V_2' could be on the far left or the far right, depending on the orientation of the square on the ground. In the van Vliet painting, we find that V_2' is located to the left of the painting, and it is very far away as you can see in Figure 5. Drawing this semi-circle accurately at the museum, where V_2' could very well be in another room, would be difficult.



`Figure 5 The vanishing point in the room next door`

Taylor [9, 1] and Lambert [7, 1] provide two alternatives, which do not require the distant V_2' vanishing point. As shown in Figure 6, Taylor suggests we draw two right isosceles triangles with hypotenuses $\overline{V_1V_1'}$ and $\overline{V_1'V_2}$ respectively then draw circles using the apexes as the centers and the legs of the triangles as the radii. The intersection of these two circles will be exactly where the intersection of the semi-circles were in the previous method, thus giving us T and d. If you're skilled at drawing right isosceles triangles and circles, this is not a bad alternative. Lambert gives us yet another alternate method, as shown in Figure 7. Again, it only requires three vanishing points, however, you have to be skilled at estimating a 45 degree angle.

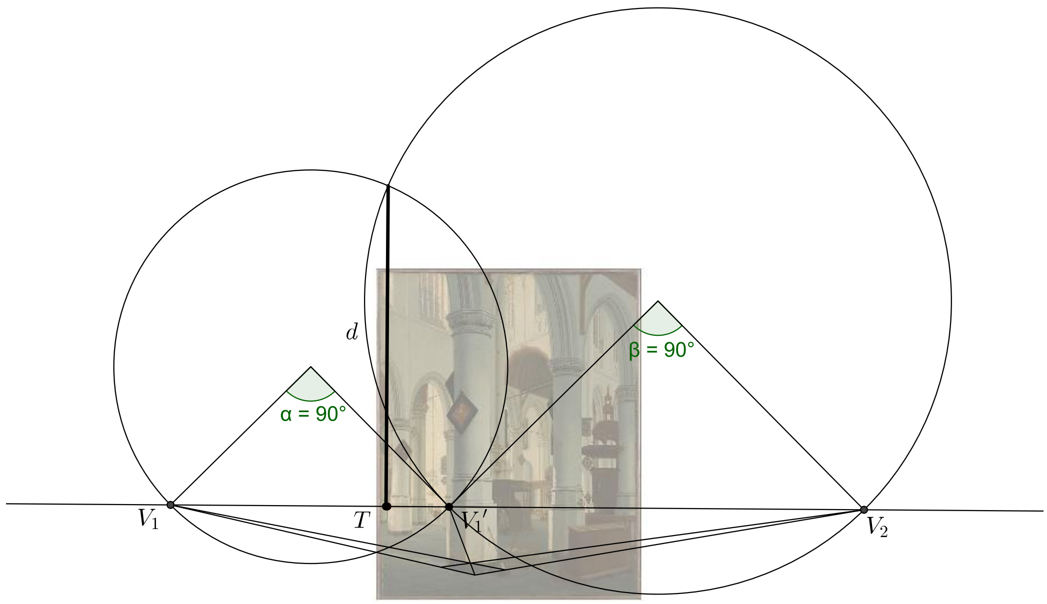


Figure 6 Taylor's method

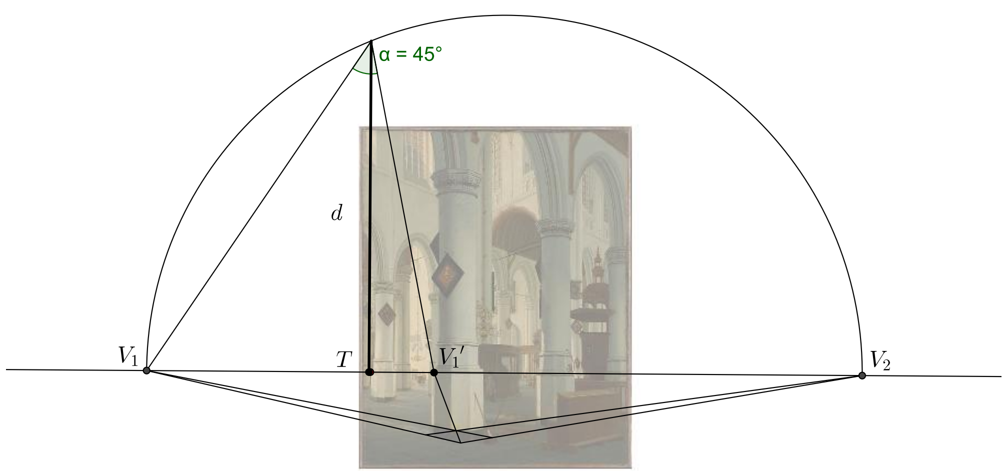


Figure 7 Lambert's method

The Algebraic Method

Since we can find T and d by finding the intersection of two circles, we can surely find an algebraic formula for T and d. In Figure 8, we superimpose a coordinate system with the horizon line as the x-axis in order to describe the semi-circles algebraically. The ratio of the distances between the principal vanishing points and the diagonal vanishing point between them become quite important, so we shall denote it as $\rho = \frac{v_L}{v_R}$, where v_L is the distance on the left (between V_1 and V_1') and v_R is the distance on the right (between V_1' and V_2).

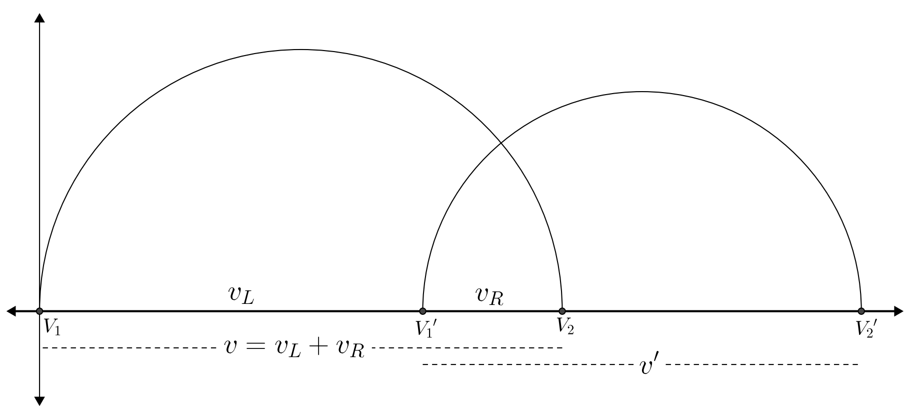


Figure 8 The algebraic method

Theorem 1.

Let t be the distance from the left-most principal vanishing point to the viewing target and d the viewing distance. Then

$$t = \frac{\rho^2 v}{\rho^2 + 1} \quad \text{and} \quad d = \frac{\rho v}{\rho^2 + 1} = \frac{t}{\rho},$$

where $\rho = \frac{v_L}{v_R}$ and $v = v_L + v_R$.

Proof.

Consider Figure 8. The equations for the semi-circles are as follows.

$$\left(x - \frac{v}{2}\right)^2 + y^2 = \left(\frac{v}{2}\right)^2 \quad \text{and} \quad \left(x - \left(v_L + \frac{v'}{2}\right)\right)^2 + y^2 = \left(\frac{v'}{2}\right)^2$$

To find t, the distance to the viewing target, we need to find the x-value of the circles' intersection point, so we subtract one equation from the other and solve for x. We find that

$$x = \frac{v_L(v_L + v')}{v_L - v_R + v'}. \tag{1}$$

We need to somehow get rid of v', so the end result is entirely in terms of v_L and v_R. As it turns out, since these four points come from a quadrangle in this particular way, a certain product of ratios of distances between the points stays constant, and this is what we use to rewrite v' in terms of v_L and v_R. This product of ratios is called the *cross ratio*.

The definition of a cross ratio requires that the distances in the product of ratios of distances indicate a direction, so that it is positive going in a chosen direction, and negative when going the opposite way. For example, given distinct points A and B on a line, we can choose the direction so that the *directed distance* $|AB|$ is positive and the directed distance $|BA|$ is negative.

The *cross ratio* of four points along a line, A, B, C and D is the following product of ratios of directed distances,

$$\times(ABCD) = \frac{|AB|}{|BC|}\frac{|CD|}{|DA|}.$$

There are some very nice properties of the cross ratio, most notably that it is invariant under projections. For more information, see [4]. The property that is most important for us here is that when the points are the principal and diagonal vanishing points of a quadrangle, then we have what's called a *harmonic set* denoted $H(V_1, V_2; V_1', V_2')$ and its cross ratio, called the *harmonic ratio*, equals -1,

$$\times(V_1V_1'V_2V_2') = \frac{|V_1V_1'|}{|V_1'V_2|}\frac{|V_2V_2'|}{|V_2'V_1|} = \frac{v_L}{v_R} \cdot \frac{v' - v_R}{-v' - v_L} = -1. \tag{2}$$

Solving for v' in (2), we find

$$v' = \frac{2v_L v_R}{v_L - v_R}. \tag{3}$$

Going back to our equation (1), substituting in (3) gives us

$$x = \frac{v_L\left(v_L + \frac{2v_L v_R}{v_L - v_R}\right)}{v_L - v_R + \frac{2v_L v_R}{v_L - v_R}} = \frac{v_L^2(v_L + v_R)}{v_L^2 + v_R^2} = \frac{(\frac{v_L}{v_R})^2(v_L + v_R)}{(\frac{v_L}{v_R})^2 + 1} = \frac{\rho^2 v}{\rho^2 + 1} = t.$$

Solving for the y-value of the intersection point gives us d.

$$y^2 = \left(\frac{v}{2}\right)^2 - \left(\frac{\rho^2 v}{\rho^2 + 1} - \frac{v}{2}\right)^2 = \frac{v^2}{4}\left(\frac{2\rho^2}{\rho^2 + 1}\right)\left(\frac{2}{\rho^2 + 1}\right) = \frac{v^2\rho^2}{(\rho^2 + 1)^2}.$$

Taking the positive solution, we get

$$d = \frac{v\rho}{\rho^2 + 1} = \frac{t}{\rho}.$$

Our algebraic method is as follows:

1. Find the vanishing points A, B and C along the horizon line.
2. Measure v_L and v_R, and calculate ρ and v.
3. Calculate $t = \frac{v\rho^2}{\rho^2+1}$ to find the viewing target T.
4. Divide t by ρ to find the viewing distance d.

Applying this to the van Vliet painting, $\rho = \frac{69.44}{103.44} \approx 0.67$, $v = 69.44 + 103.44 = 172.88$. Thus $t \approx 53.71$ cm and $d \approx 80.00$ cm. In Figure 9, we see where this T is located on the picture plane, and we see that it nearly coincides with the T found using the geometric method. The small difference of approximately 0.34 cm is due to round-off error. We can make the calculation easier, by approximating $\rho \approx 2/3$ and $v \approx 173$, and the simple calculation $173 \cdot 4/13 \approx 53.23$ is still less than a centimeter off.

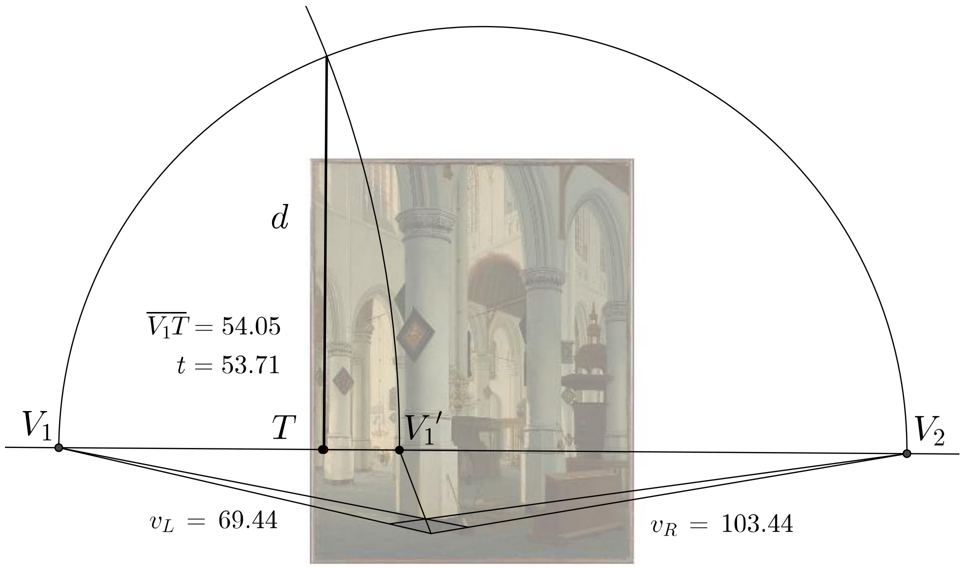


Figure 9 The algebraic method

We're certainly not the first to give an algebraic formula. Greene [6] came up with a similar formula for t and d, translated below into our notation ,

$$t = \frac{vv_L^2}{v^2 - 2vv_L + 2v_L^2} \ , \ d = \frac{[vv_L^2(v^3 - 2v^2v_L + vv_L^2)]^{1/2}}{v^2 - 2vv_L + 2v_L^2}.$$

Instead of ρ and v, he used v_L and v. Although we can simplify his d to $d = \frac{vv_L(v-v_L)}{v^2-2vv_L+2v_L^2}$, this formula is considerably harder to remember and harder to use at the museum.

The Perspective Slope Method

We come now to our new method, which we call the perspective slope method. Earlier, we mentioned that the cross ratio of a harmonic set equals -1. Let's see what happens when we replace the diagonal vanishing point V_1' with T.

$$\times(V_1TV_2V_2') = \frac{|V_1T|}{|TV_2|}\frac{|V_2V_2'|}{|V_2'V_1|} = \frac{t}{t - v_L} \cdot \frac{v' - v_R}{v' + v_L}. \qquad (4)$$

By (2), $\frac{v'-v_R}{v'+v_L} = -\frac{1}{\rho}$. So using this substitution along with the formula for t in (4), we find

$$\times(V_1TV_2V_2') = \frac{\frac{\rho^2 v}{\rho^2+1}}{v - \frac{\rho^2 v}{\rho^2+1}} \frac{-1}{\rho} = \frac{\rho^2 v}{\rho^2 v + v - \rho^2 v} \frac{-1}{\rho} = -\rho.$$

This is rather nice. So how might we use it?

In Figure 10, we have a special harmonic set, with lengths $|V_1V_1'| = 1/3$, $|V_1V_2| = 1/2$, and $|V_1V_2'| = 1$ (by the way, this is why it's called a harmonic set, because of its relationship to the harmonic sequence!). In this case, $v = 1/2$, $\rho = 2$ and we have

$$t = \frac{2^2\left(\frac{1}{2}\right)}{2^2+1} = \frac{2}{5}.$$

If we think of $PQRS$ as the perspective image of a square on the ground, we can use the square to form a coordinate grid with line PQ as the image of the horizontal axis and PS the image of the vertical axis. Then line QS is the image of a line with slope -1 in this coordinate system and it vanishes at V_1'. We also have $\times(V_1V_1'V_2V_2') = -1$. Now we draw the image of a line with slope -2. To draw this, we draw the next square back as shown in Figure 10, remembering that lines parallel in the real world not parallel to the canvas converge to a common point (including the diagonals of the squares!). This line vanishes at T, and $\times(V_1TV_2V_2') = -\rho = -2$! It's no coincidence that the slope of this diagonal line and the cross ratio equal.

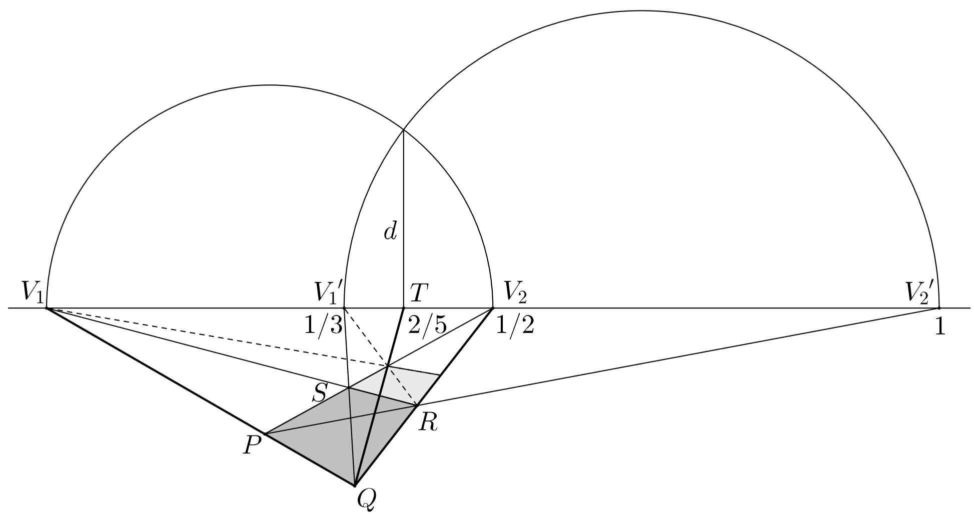


Figure 10 Perspective slope

THEOREM 2.

Let $PQRS$ be a quadrangle with associated harmonic set $H(V_1, V_2; V_1', V_2')$. Assuming that $PQRS$ is a square in perspective with the perspective coordinate grid set up as described above, a line with slope m in the perspective coordinate grid will vanish at a point M such that $\times(V_1 M V_2 V_2') = m$.

Proof.

We first look at the one point perspective situation, as in Figure 11. ℓ will vanish at M located d/m away from V_2. This is clear from the top view, but details are found in [5]. Hence, the cross ratio will be

$$\times(V_1 M V_2 V_2') = \frac{|V_1 M|}{|M V_2|}\frac{|V_2 V_2'|}{|V_2' V_1|} = \frac{\infty}{-d/m}\frac{d}{-\infty} = m.$$

Since the cross ratio is a projective invariant, this will also hold in two point perspective.

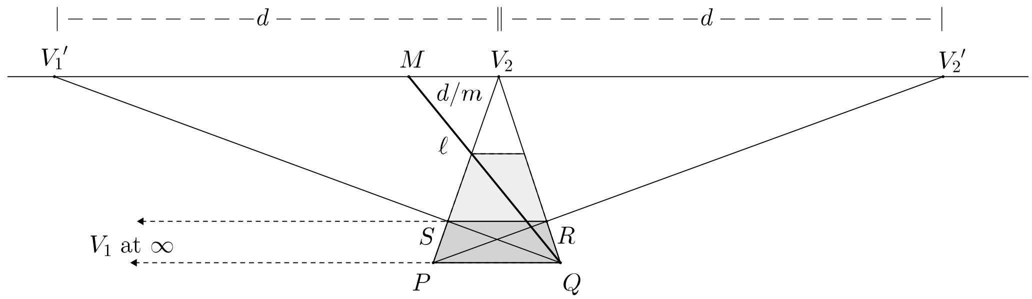


Figure 11 One point perspective situation

So in order to find T, we need only calculate $\rho = \frac{v_L}{v_R}$, draw the line with slope $-\rho$ and its vanishing point will be T! Once you determine the location of T and measure $t = |V_1T|$, $d = t/\rho$.

So our final method is as follows:

1. Find the vanishing points V_1, V_2 and V_1' along the horizon line.
2. Measure $|AB|$ and $|BC|$, and calculate $\rho = \frac{v_L}{v_R}$.
3. Use your perspective drawing skills to draw a line with slope $-\rho$ in perspective, and find its vanishing point. This is T.
4. Divide $t = |V_1T|$ by ρ to find the viewing distance d.

We approximate 0.67 by 2/3, which makes finding the slope a bit easier. One way is to draw a 2×3 grid, as shown in Figure 12. Despite our rough estimation, it does a pretty good job!

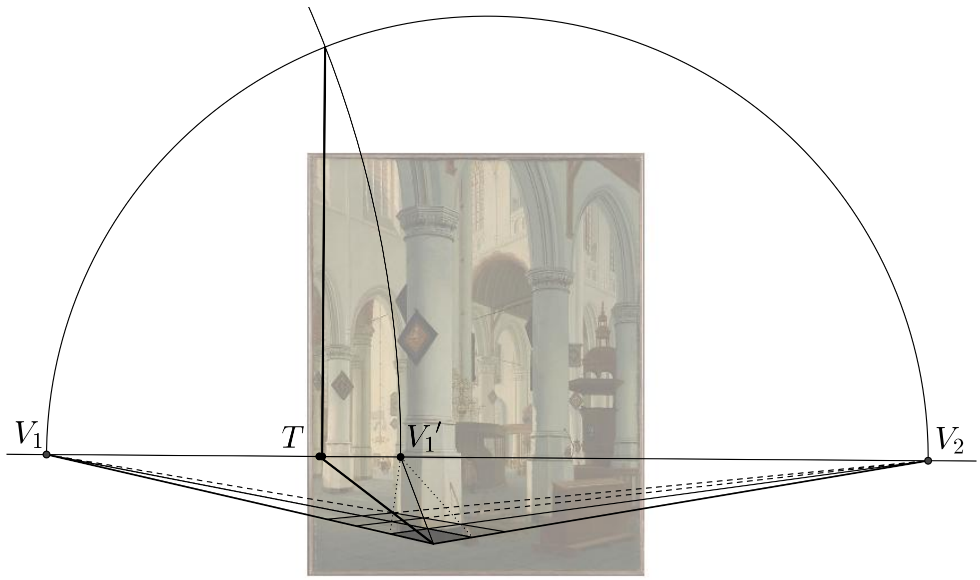


Figure 12 The perspective slope method

The viewing distance is then roughly 3/2 of t.

The advantage of this method is that the calculations are much simpler and it's fun to use your perspective drawing skills! If there is a tiled floor, this is a very easy method to use.

Each method has its advantages and disadvantages. But being able to find the viewing position is invaluable; if you try this with the van Vliet painting, the arches suddenly soar overhead and you can feel the spaciousness of the centuries old church. We hope that you will use one of these methods the next time you run across a work in two point perspective to magically transport yourself into its world!

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Referee's Appendix

Here, we give the details of the computation of the algebraic method.

Theorem 1.

Let t be the distance from the left-most vanishing point to the viewing target and d the viewing distance. Then

$$t = \frac{v\rho^2}{\rho^2 + 1} \quad \text{and} \quad d = \frac{v\rho}{\rho^2 + 1},$$

where $\rho = \frac{v_L}{v_R}$ and $v = v_L + v_R$.

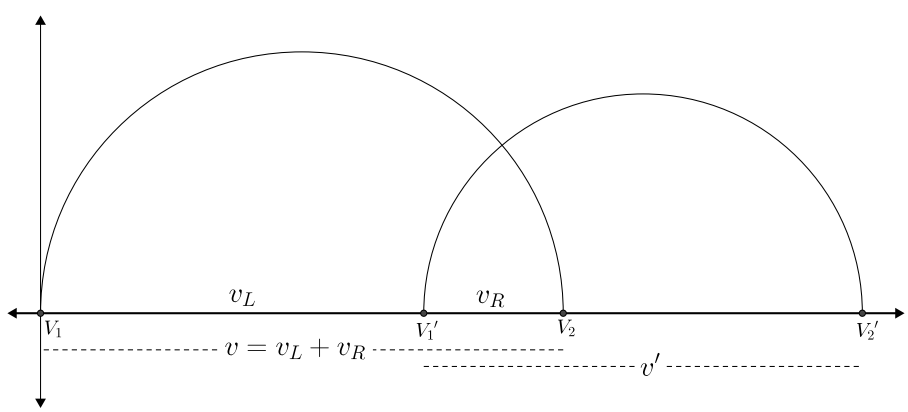

Figure 8 The algebraic method

Consider Figure 8. The equations for the semi-circles are as follows.

$$\left(x - \frac{v}{2}\right)^2 + y^2 = \left(\frac{v}{2}\right)^2 \quad \text{and} \quad \left(x - \left(v_L + \frac{v'}{2}\right)\right)^2 + y^2 = \left(\frac{v'}{2}\right)^2$$

To find t, the distance to the viewing target, we need to find the x-value of the circles' intersection point, so we subtract one equation from the other and solve for x.

$$\begin{aligned}
\left(x-\frac{v}{2}\right)^2-\left(x-\left(v_L+\frac{v'}{2}\right)\right)^2 &= \left(\frac{v}{2}\right)^2-\left(\frac{v'}{2}\right)^2\\
\left(\left(x-\frac{v}{2}\right)+\left(x-\left(v_L+\frac{v'}{2}\right)\right)\right)\left(\left(x-\frac{v}{2}\right)-\left(x-\left(v_L+\frac{v'}{2}\right)\right)\right) &= \left(\frac{v}{2}\right)^2-\left(\frac{v'}{2}\right)^2\\
\left(2x-\left(\frac{v}{2}+v_L+\frac{v'}{2}\right)\right)\left(-\frac{v}{2}+v_L+\frac{v'}{2}\right) &= \frac{1}{4}(v^2-v'^2)\\
\left(4x-(v+2v_L+v')\right)(-v+2v_L+v') &= v^2-v'^2.
\end{aligned}$$

Isolating x and simplifying, we obtain

$$\begin{aligned}
x &= \frac{1}{4}\left(\frac{v^2-v'^2}{-v+2v_L+v'}+v+2v_L+v'\right)\\
&= \frac{v_L^2+v_Lv'}{-v+2v_L+v'}\\
&= \frac{v_L(v_L+v')}{v_L-v_R+v'}. \qquad (5)
\end{aligned}$$

To get rid of v', we use the harmonic ratio,

$$\times(V_1V_1'V_2V_2') = \frac{|V_1V_1'|}{|V_1'V_2|}\frac{|V_2V_2'|}{|V_2'V_1|} = \frac{v_L}{v_R}\cdot\frac{v'-v_R}{-v'-v_L} = -1. \qquad (6)$$

Solving for v' in (6), we find

$$\begin{aligned}
v_Lv'-v_Lv_R &= v_Rv'+v_Lv_R\\
v'(v_L-v_R) &= 2v_Lv_R\\
v' &= \frac{2v_Lv_R}{v_L-v_R}. \qquad (7)
\end{aligned}$$

Going back to our equation (5), substituting (7) gives us

$$\begin{aligned}
x &= \frac{v_L(v_L + v')}{v_L - v_R + v'} \\
&= \frac{v_L\left(v_L + \frac{2v_L v_R}{v_L - v_R}\right)}{v_L - v_R + \frac{2v_L v_R}{v_L - v_R}} \\
&= \frac{v_L^2(v_L + v_R)}{v_L^2 + v_R^2} \\
&= \frac{(\frac{v_L}{v_R})^2(v_L + v_R)}{(\frac{v_L}{v_R})^2 + 1} \\
&= \frac{\rho^2 v}{\rho^2 + 1} = t.
\end{aligned}$$

Solving for the y value of the intersection point gives us d.

$$\begin{aligned}
y^2 &= \left(\frac{v}{2}\right)^2 - \left(\frac{\rho^2 v}{\rho^2 + 1} - \frac{v}{2}\right)^2 \\
&= \left(\frac{v}{2}\right)^2 \left(1 - \left(\frac{\rho^2 - 1}{\rho^2 + 1}\right)^2\right) \\
&= \frac{v^2}{4}\left(1 - \left(\frac{\rho^2 - 1}{\rho^2 + 1}\right)\right)\left(1 + \left(\frac{\rho^2 - 1}{\rho^2 + 1}\right)\right) \\
&= \frac{v^2}{4}\left(\frac{2\rho^2}{\rho^2 + 1}\right)\left(\frac{2}{\rho^2 + 1}\right) \\
&= \frac{v^2 \rho^2}{(\rho^2 + 1)^2}.
\end{aligned}$$

Taking the positive solution, we get $d = \frac{v\rho}{\rho^2+1}$.

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